

Mapping-class groups of 3-manifolds in canonical quantum gravity

Domenico Giulini

This contribution is dedicated to Rafael Sorkin on the occasion of his 60th birthday

Abstract. Mapping-class groups of 3-manifolds feature as symmetry groups in canonical quantum gravity. They are an obvious source through which topological information could be transmitted into the quantum theory. If treated as gauge symmetries, their inequivalent unitary irreducible representations should give rise to a complex superselection structure. This highlights certain aspects of spatial diffeomorphism invariance that to some degree seem physically meaningful and which persist in all approaches based on smooth 3-manifolds, like geometrodynamics and loop quantum gravity. We also attempt to give a flavor of the mathematical ideas involved.

Mathematics Subject Classification (2000). Primary 83C45; Secondary 57N10.

Keywords. Canonical Quantum Gravity, 3-manifolds, mapping classes.

1. Some facts about Hamiltonian General Relativity

1.1. Introduction

As is well known, Einstein's field equations for General Relativity can be cast into the form of a constrained Hamiltonian system. The unreduced phase space is then given by the cotangent bundle over the space $\text{Riem}(\Sigma)$ of Riemannian metrics on a 3-manifold Σ . This phase space is coordinatized by (q, p) , where q is a Riemannian metric on Σ and p is a section in the bundle of symmetric contravariant tensors of rank two and density-weight one over Σ .

The relation of these objects to the description of a solution to Einstein's equations in terms of a four-dimensional globally hyperbolic Lorentzian manifold

Contribution to the proceedings of the workshop on *mathematical and physical aspects of quantum gravity* held in Blaubeuren/Germany from July 28th to August 1st 2005. I sincerely thank Bertfried Fauser and Jürgen Tolksdorf for organizing Blaubeuren II and inviting me to it.

(g, M) is as follows: let Σ be a Cauchy surface in M and $q := g|_{T(\Sigma)}$ its induced metric (first fundamental form). Let n be the normal field to Σ , where $g(n, n) = -1$, i.e. we use the ‘mostly plus’ signature convention where timelike vectors have a negative g -square. Let D and ∇ be the Levi-Civita covariant derivatives on (q, Σ) and (g, M) respectively. Then for any vector X tangent to Σ and any vector field \tilde{Y} on M whose restriction Y to Σ is tangential to Σ , we have

$$\nabla_X \tilde{Y} = D_X Y + n K(X, Y), \quad (1.1)$$

where $n K(X, Y)$ represents the normal component of $\nabla_X \tilde{Y}$. It is easy to see that K is a symmetric covariant tensor of rank 2 on Σ , also called the extrinsic curvature (or second fundamental form) of Σ in M . The canonical momentum field p can now be expressed in terms of these data:

$$p = \sqrt{\det(q)} (K - q \operatorname{Tr}_q(K))^\sharp. \quad (1.2)$$

Here \sharp denotes the isomorphism $T^*(M) \rightarrow T(M)$ induced by q (‘index raising’), extended to tensors of all ranks.

Einstein’s equations in Hamiltonian form now decompose into two evolution equations (of six independent component equations each):

$$\dot{q} = F_1(q, p; \alpha, \beta; \phi), \quad (1.3a)$$

$$\dot{p} = F_2(q, p; \alpha, \beta; \phi), \quad (1.3b)$$

and two equations without time derivatives in q and p (of one and three independent component equations respectively), thereby implying constraints on the initial data:

$$C_s(q, p; \phi) := G_q(p, p) - \sqrt{\det(q)} (S(q) - 2\Lambda) + 2\rho_m(\phi, q) = 0, \quad (1.4a)$$

$$C_v(q, p; \phi) := -2\operatorname{div}_q p + j_m(\phi, q) = 0. \quad (1.4b)$$

These are referred to as the scalar and the vector (or diffeomorphism) constraints respectively.

The meanings of the symbols in (1.3) and (1.4) are as follows: $F_{1,2}$ are local functionals whose explicit forms need not interest us at this moment. α and β are a scalar function and a vector field on Σ respectively (the ‘lapse’ and ‘shift’ function) which are not determined by the equations of motion but which one needs to specify by hand. They represent the four free functions out of the ten component functions $g_{\mu\nu}$ which are *not* determined by the equations of motion due to spacetime-diffeomorphism invariance. $S(q)$ is the scalar curvature (Ricci scalar) of (q, Σ) and div_q denotes the covariant divergence with respect to the Levi Civita derivative on (q, Σ) , i.e. in components $(\operatorname{div}_q p)^b = D_a p^{ab}$. The symbol ϕ collectively represents possible matter fields. ρ_m and j_m are respectively the scalar and vector densities of weight one for the energy and momentum density of the matter. As usual, Λ is the cosmological constant. Finally, G_q is a bilinear form, the so-called DeWitt metric, that maps a pair of symmetric contravariant 2nd-rank

tensors of density weight one to a scalar density of weight one. In components one has

$$\begin{aligned} G_q(p_1, p_2) &= [\det(q_{ab})]^{-1/2} \frac{1}{2} (q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd}) p_1^{ab} p_2^{cd} \\ &= [\det(q_{ab})]^{-1/2} (p_1^{ab} p_{2ab} - \frac{1}{2} p_{1a}^a p_{2b}^b). \end{aligned} \quad (1.5)$$

Pointwise (on Σ) it defines a Lorentzian metric of signature $(1, 5)$ (the ‘negative direction’ being the trace mode) on the six dimensional space of symmetric 2nd-rank tensor densities, a discussion of which may be found in [10]. Some relevant aspects of the infinite-dimensional geometry that is obtained by integrating $G_q(p_1, p_2)$ over Σ (the so-called ‘Wheeler-DeWitt metric’) are discussed in [28].

1.2. Topologically closed Cauchy surfaces

If Σ is closed (compact without boundary) the constraints (1.4) actually generate all of the dynamical evolution (1.3). That is, we can write

$$F_1 = \frac{\delta H}{\delta p}, \quad (1.6a)$$

$$F_2 = -\frac{\delta H}{\delta q}, \quad (1.6b)$$

where

$$H[q, p; \alpha, \beta; \phi] = \int_{\Sigma} \alpha C_s(q, p; \phi) + \int_{\Sigma} \beta \cdot C_v(q, p; \phi). \quad (1.7)$$

Here $\beta \cdot C_v$ denote the natural pairing between a vector (β) and a one-form of density weight one (C_v). This means that the entire dynamical evolution is generated by constraints. These constraints form a first-class system, which means that the Poisson bracket of two of them is again a linear combination (generally with phase-space dependent coefficients) of the constraints. Writing

$$C_s(\alpha) := \int_{\Sigma} \alpha C_s, \quad \text{and} \quad C_v(\beta) := \int_{\Sigma} \beta \cdot C_v \quad (1.8)$$

we have

$$\{C_s(\alpha), C_s(\alpha')\} = C_v(\alpha(d\alpha')^{\sharp} - \alpha'(d\alpha)^{\sharp}), \quad (1.9a)$$

$$\{C_v(\beta), C_s(\alpha)\} = C_s(\beta \cdot d\alpha), \quad (1.9b)$$

$$\{C_v(\beta), C_v(\beta')\} = C_v([\beta, \beta']). \quad (1.9c)$$

In passing we remark that (1.9c) says that the vector constraints form a Lie-subalgebra which, however, according to (1.9b), is *not* an ideal. This means that the flows generated by the scalar constraints are not tangential to the constraint-hypersurface that is determined by the vanishing of the vector constraints, except for the points where the constraint-hypersurfaces for the scalar and vector constraints intersect. This means that generally one cannot reduce the constraints in steps: *first* the vector constraints and *then* the scalar constraints, simply because the scalar constraints do not act on the solution space of the vector constraints.

This difficulty clearly persists in any implementation of (1.9) as operator constraints in canonical quantum gravity.

According to the orthodox theory of constrained systems [11, 48], all motions generated by the *first class* constraints should be interpreted as gauge transformations, i.e. be physically unobservable.¹ In other words, states connected by a motion that is generated by first-class constraints are to be considered as *physically identical*.

The conceptual question of how one should interpret the fact that all evolution is pure gauge is known as the *problem of time* in classical and also in quantum general relativity. It is basically connected with the constraints $C_s(\alpha)$, since their Hamiltonian flow represents a change on the canonical variables that corresponds to the motion of the hypersurface Σ in M in normal direction. For a detailed discussion see Section 5.2 in [53].

In contrast, the meaning of the flow generated by the constraints $C_v(\beta)$ is easy to understand: it just corresponds to an infinitesimal diffeomorphism within Σ . Accordingly, its action on a local² phase-space function $F[q, p](x)$ is just given by its Lie derivative:

$$\{F, C_v(\beta)\} = L_\beta F. \quad (1.10)$$

Hence the gauge group generated by the vector constraints is the identity component $\text{Diff}^0(\Sigma)$ of the diffeomorphism group $\text{Diff}(\Sigma)$ of Σ . Note that this is true despite the fact that $\text{Diff}(\Sigma)$ is only a Fréchet Lie group and that, accordingly, the exponential map is not surjective on any neighborhood of the identity (cf. [18]). The point being that $\text{Diff}^0(\Sigma)$ is simple (cf. [64]) and that the subgroup generated³ by the image of the exponential map is clearly a non-trivial normal subgroup of, and hence equivalent to, $\text{Diff}^0(\Sigma)$.

What about those transformations in $\text{Diff}(\Sigma)$ which are not in the identity component (i.e. the so called ‘large’ diffeomorphisms)? Are they, too, to be looked at as pure gauge transformations, or are they physically meaningful (observable) symmetries? Suppose we succeeded in constructing the reduced phase space with

¹In [11] Dirac proposed this in the form of a conjecture. It has become the orthodox view that is also adopted in [48]. There are simple and well known—though rather pathological—counterexamples (e.g. [48] § 1.2.2 and § 1.6.3). The conjecture can be proven under the hypothesis that there are no *ineffective* constraints, i.e. whose Hamiltonian vector fields vanish on the constraint hypersurface (e.g. [8]; also [48] § 3.3.2). For further discussion of this rather subtle point see [34] and [35]. Note however that these issues are only concerned with the algebraic form in which the constraints are delivered by the formalism. For example, if $\phi(q, p) = 0$ is effective (i.e. $d\phi|_{\phi^{-1}(0)} \neq 0$) then $\phi^2(q, p) = 0$ is clearly ineffective, even though it defines the *same* constraint subset in phase space. In a proper geometric formulation the algebra of observables just depends on this constraint subset: define the ‘gauge Poisson-algebra’, Gau , by the set of all smooth functions that vanish on this set (it is clearly an ideal with respect to pointwise multiplication, but not a Lie-ideal). Then take as algebra of physical observables the quotient of the Lie idealizer of Gau (in the Poisson algebra of, say, smooth functions on unconstrained phase space) with respect to Gau . See e.g. [31] for more details.

²‘Local’ meaning that the real-valued function F on Σ depends on $x \in \Sigma$ through the values of q and p as well their derivatives up to *finite* order at x .

³The subgroup ‘generated’ by a set is the subgroup of all finite products of elements in this set.

respect to $\text{Diff}^0(\Sigma)$, we would then still have a residual non-trivial action of the discrete group

$$\mathcal{G}(\Sigma) := \text{Diff}(\Sigma)/\text{Diff}^0(\Sigma) =: \pi_0(\text{Diff}(\Sigma)). \quad (1.11)$$

Would we then address as physical *observables* only those functions on phase space which are invariant under $\mathcal{G}(\Sigma)$? The answer to this questions may well depend on the specific context at hand. But since $\mathcal{G}(\Sigma)$ is generically a non-abelian and infinite group, the different answers will have significant effect on the size and structure of the space of physical states and observables. A 2+1 dimensional model where this has been studied in some detail is presented in [25].

Reducing the configuration space $\text{Riem}(\Sigma)$ by the action of $\text{Diff}(\Sigma)$ leads to what is called ‘superspace’⁴:

$$\mathcal{S} = \text{Riem}(\Sigma)/\text{Diff}(\Sigma). \quad (1.12)$$

It can be given the structure of a stratified manifold [14], where the nested singular sets are labeled by the isotropy groups of $\text{Diff}(\Sigma)$ (i.e. the singular sets are the geometries with non-trivial isotropy group and nested according to the dimensionality of the latter.)

There is a natural way to resolve the singularities of \mathcal{S} [15], which can be described as follows: pick a point $\infty \in \Sigma$ (we shall explain below why the point is given that name) and consider the following subgroups of $\text{Diff}(\Sigma)$ that fix ∞ and frames at ∞ respectively:

$$\text{Diff}_\infty(\Sigma) := \{\phi \in \text{Diff}(\Sigma) : \phi(\infty) = \infty\}, \quad (1.13a)$$

$$\text{Diff}_F(\Sigma) := \{\phi \in \text{Diff}_\infty(\Sigma) : \phi_*|_\infty = \text{id}|_{T_\infty(\Sigma)}\}. \quad (1.13b)$$

The resolved Superspace, \mathcal{S}_F , is then isomorphic to⁵

$$\mathcal{S}_F := \text{Riem}(\Sigma)/\text{Diff}_F(\Sigma). \quad (1.14)$$

The point is that $\text{Diff}_F(\Sigma)$ acts freely on $\text{Riem}(\Sigma)$ due to the fact that diffeomorphisms that fix the frames at one point cannot contain non-trivial isometries.⁶ This, as well as the appropriate slicing theorems for the surjection $\text{Riem}(\Sigma) \rightarrow \mathcal{S}_F$ (which already holds for the action of $\text{Diff}(\Sigma)$, see [14] and references therein) then establish a manifold structure of \mathcal{S}_F . In fact, we have a principle bundle

$$\text{Diff}_F(\Sigma) \xrightarrow{i} \text{Riem}(\Sigma) \xrightarrow{p} \mathcal{S}_F. \quad (1.15)$$

The contractibility of $\text{Riem}(\Sigma)$ (which is an open convex cone in a topological vector space) implies that $\text{Riem}(\Sigma)$ is a universal classifying bundle and \mathcal{S}_F a universal classifying space for the group $\text{Diff}_F(\Sigma)$. It also implies, via the long exact homotopy sequence for (1.15), that

$$\pi_n(\text{Diff}_F(\Sigma)) \cong \pi_{n+1}(\mathcal{S}_F) \quad \text{for } n \geq 0. \quad (1.16)$$

⁴This notion is independent to similarly sounding ones in the theory of supersymmetric field theories

⁵The isomorphism is non canonical since we had to select a point $\infty \in \Sigma$.

⁶To see this, assume $\phi \in \text{Diff}_F(\Sigma)$ is an isometry of (q, Σ) , where Σ is connected. The set of fixed points is clearly closed. It is also open, as one readily sees by using the exponential map.

Recall that π_0 of a topological group G is a group (this is not true for arbitrary topological spaces) which is defined by G/G^0 , where G^0 is the identity component. Hence we have

$$\mathcal{G}_F(\Sigma) := \text{Diff}_F(\Sigma)/\text{Diff}_F^0(\Sigma) =: \pi_0(\text{Diff}_F(\Sigma)) \cong \pi_1(\mathcal{S}_F). \quad (1.17)$$

In this way we recognize the mapping-class group for frame-fixing diffeomorphisms of Σ as the fundamental group of the singularity-resolved $\text{Diff}(\Sigma)$ -reduced configuration space of canonical gravity. Next to (1.11) and (1.17) we also introduce the analogous mapping class groups for point-fixing diffeomorphisms:

$$\mathcal{G}_\infty(\Sigma) := \text{Diff}_\infty(\Sigma)/\text{Diff}_\infty^0(\Sigma) =: \pi_0(\text{Diff}_\infty(\Sigma)). \quad (1.18)$$

1.3. Topologically open Cauchy surfaces

So far we assumed Σ to be closed. This is the case of interest in cosmology. However, in order to model isolated systems, one is interested in 3-manifolds Σ' with at least one asymptotically flat end, where here we restrict to the case of one end only. The topological implication behind ‘asymptotical flatness’ is simply the requirement that the one-point compactification $\Sigma = \Sigma' \cup \infty$ (here ∞ is the point added) be a manifold. This is equivalent to the existence of a compact set $K \subset \Sigma'$ such that $\Sigma' - K$ is homeomorphic to $S^2 \times \mathbb{R}$ (i.e. $\mathbb{R}^3 - \text{ball}$).

The analytic expressions given in Section 1.1 made no reference to whether Σ is open or closed. In particular, the constraints are still given by (1.4). However, in the open case it is not true anymore that the dynamical evolution is entirely driven by the constraints, as in (1.6) and (1.7). Rather, we still have (1.6) but must change (1.7) to⁷

$$\begin{aligned} H[q, p; \alpha, \beta; \phi] = \lim_{R \rightarrow \infty} \left\{ \int_{B_R} \alpha C_s(q, p; \phi) + \int_{B_R} \beta \cdot C_v(q, p; \phi) \right. \\ \left. + \int_{S_R} \mathcal{E}(\alpha; q, p) + \int_{S_R} \mathcal{M}(\beta; q, p) \right\}. \end{aligned} \quad (1.19)$$

Here B_R is a sequence of compact sets, labeled by their ‘radius’ R , so that $R' > R$ implies $B_{R'} \supset B_R$ and $\lim_{R \rightarrow \infty} B_R = \Sigma'$. S_R is equal to the boundary ∂B_R , which we assume to be an at least piecewise differentiable embedded 2-manifold in Σ' . \mathcal{E} and \mathcal{M} are the fluxes for energy and linear momentum if asymptotically for large R the lapse function α assumes the constant value 1 and β approaches a translational Killing vector. Correspondingly, if β approaches a rotational Killing vector, we obtain the flux for angular momentum (see [6] for the analytic expressions in case of pure gravity). Since the constraints (1.9) must still be satisfied as part of Einstein’s equations, we see that ‘on shell’ the Hamiltonian (1.19) is a sum of surface integrals. Note also that even though the surface integrals do not explicitly depend on the matter variables ϕ , as indicated in (1.19), there is an implicit dependence through the requirement that (q, p, ϕ) satisfy the constraints (1.9). This must be so since

⁷Here we neglect other surface integrals that arise in the presence of gauge symmetries other than diffeomorphism invariance whenever globally charged states are considered.

these surface integrals represent the total energy and momentum of the system, including the contributions from the matter.

Let us consider the surface integral associated with the spatial vector field β . It is given by

$$P(q, p; \beta) := 2 \lim_{R \rightarrow \infty} \left\{ \int_{S_R} p(n^b, \beta^b) \right\} \quad (1.20)$$

where n is the outward pointing normal of S_R in Σ' and $n^b := q(n, \cdot)$ etc. It is precisely minus the surface integral that emerges by an integration by parts from the second integral on the right hand side of (1.7) and which obstructs functional differentiability with respect to p . The addition of (1.20) to (1.7) just leads to a cancellation of both surface integrals thereby restoring functional differentiability for non-decaying β . This is precisely what was done in (1.19). Conversely, this shows that the constraint $C_v(\beta)$ (cf. (1.8)) only defines a Hamiltonian vector field if β tends to zero at infinity. Hence the constraints C_v only generate asymptotically trivial diffeomorphisms. The rate of this fall-off is of crucial importance for detailed analytical considerations, but is totally unimportant for the topological ideas we are going to present. For our discussion it is sufficient to work with $\Sigma = \Sigma' \cup \infty$. In particular, the group of spatial diffeomorphisms generated by the constraints may again be identified with $\text{Diff}_F^0(\Sigma)$. This is true since we are only interested in homotopy invariants of the diffeomorphism group⁸ and the group of diffeomorphisms generated by the constraints is homotopy equivalent to $\text{Diff}_F^0(\Sigma)$, whatever the precise fall-off conditions for the fields on Σ' are. Moreover, the full group of diffeomorphisms, $\text{Diff}(\Sigma')$, is homotopy equivalent to $\text{Diff}_\infty(\Sigma)$.⁹

To sum up, the configuration space topology in Hamiltonian General Relativity is determined by the topology of $\text{Diff}_F(\Sigma)$, where Σ is a closed 3-manifold. This is true in case the Cauchy surface is Σ and also if the Cauchy surface is open with one regular end, in which case Σ is its one-point compactification. In particular, the fundamental group of configuration space is isomorphic to the mapping-class groups (1.17). This is the object we shall now focus attention on. It has an obvious interest for the quantization program: for example, it is well known from elementary Quantum Mechanics that the inequivalent irreducible unitary representations of the fundamental group of the classical configuration space (the domain of the Schrödinger function) label inequivalent quantum sectors; see e.g. [27] and references therein. Even though in field theory it is not true that the classical configuration space is the proper functional-analytic domain of the Schrödinger state-functional, it remains true that its fundamental group— $\mathcal{G}_F(\Sigma)$ in our case—acts as group of (gauge) symmetries on the space of quantum states. Hence one is naturally interested in the structure and the representations of such groups. Some applications of these concepts in quantum cosmology and 2+1 quantum gravity

⁸ We already alert to the fact that homotopy invariants of the groups of homeomorphisms or diffeomorphisms of a 3-manifold Σ are topological invariants of Σ but *not* necessarily also homotopy invariants of Σ . We will come back to this below.

⁹To see this one needs, in particular, to know that $\text{Diff}(S^2)$ is homotopy equivalent to $SO(3)$ [76].

may be found in [32] [25] respectively. The whole discussion on θ -sectors¹⁰ in quantum gravity started in 1980 with the seminal paper [19] by John Friedman and Rafael Sorkin on the possibility of spin-1/2 states in gravity. We will discuss this below. Further discussion of mapping-class groups as symmetry groups in canonical quantum gravity and their physical relevance are given in [51, 77, 79, 2].

In analogy to standard gauge theories of Yang-Mills type, one may speculate that the higher homotopy groups of $\text{Diff}_F(\Sigma)$ are also of physical significance, e.g. concerning the question of various types of anomalies [1, 69]. Such groups may be calculated for large classes of prime 3-manifolds [26] (the concept of a prime manifold is explained below), but not much seems to be known in the general reducible case.¹¹

As already pointed out (cf. footnote 8), the homotopy invariants of $\text{Diff}_F(\Sigma)$ are topological but not necessarily also homotopy invariants of Σ (cf. [61]). For example, if Σ is a spherical space form, that is $\Sigma \cong S^3/G$ with finite $G \subset SO(4)$, the mapping class group $\mathcal{G}_F(\Sigma)$ often fully characterizes Σ and can even sometimes distinguish two non-homeomorphic Σ 's which are homotopy equivalent. The latter happens in case of lens spaces, $L(p, q)$, where generally p and q denote any pair of coprime integers. Their mapping-class groups¹² are as follows: 1.) $\mathcal{G}_F(\Sigma) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ if $q^2 \equiv 1 \pmod{p}$ and $q \not\equiv \pm 1 \pmod{p}$, 2.) $\mathcal{G}_F(\Sigma) \cong \mathbb{Z}_4$ if $q^2 \equiv -1 \pmod{p}$, and 3.) $\mathcal{G}_F(\Sigma) \cong \mathbb{Z}_2$ in the remaining cases; see Table II. p. 581 in [86]. On the other hand, it is known that two lens spaces $L(p, q)$ and $L(p, q')$ are homeomorphic iff¹³ $q' \equiv \pm q \pmod{p}$ or $qq' \equiv \pm 1 \pmod{p}$ [71] and homotopy equivalent iff qq' or $-qq'$ is a quadratic residue mod p , i.e. iff $qq' \equiv \pm n^2 \pmod{p}$ for some integer n [85]. For example, this implies that $L(15, 1)$ and $L(15, 4)$ are homotopy equivalent but not homeomorphic and that the mapping class group of $L(15, 1)$ is \mathbb{Z}_2 whereas that of $L(15, 4)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$. Further distinctions can be made using the fundamental group of $\text{Diff}_F(\Sigma)$. See the table on p. 922 in [24] for more information.

¹⁰ θ symbolically stands for the parameters that label the equivalence classes of irreducible unitary representations. This terminology is borrowed from QCD, where the analog of $\text{Diff}_F(\Sigma)$ is the group of asymptotically trivial $SU(3)$ gauge-transformations, whose associated group of connected components—the analog of $\text{Diff}_F(\Sigma)/\text{Diff}_F^0(\Sigma)$ —is isomorphic to $\pi_3(SU(3)) \cong \mathbb{Z}$. The circle-valued parameter θ then just labels the equivalence classes of irreducible unitary representations of that \mathbb{Z} .

¹¹For example, for connected sums of three or more prime manifolds, the fundamental group of the group of diffeomorphisms is not finitely generated [52].

¹²A special feature of lens spaces is that $\mathcal{G}(\Sigma) \cong \mathcal{G}_\infty(\Sigma) \cong \mathcal{G}_F(\Sigma)$, where $\Sigma = L(p, q)$. These groups are also isomorphic to $\text{Isom}(\Sigma)/\text{Isom}^0(\Sigma)$, where Isom denotes the group of isometries with respect to the metric of constant positive curvature. The property $\mathcal{G}(\Sigma) \cong \text{Isom}(\Sigma)/\text{Isom}^0(\Sigma)$ is known to hold for many of the spherical space forms; an overview is given in [24] (see the table on p. 922). It is a weakened form of the *Hatcher Conjecture* [39], which states that the inclusion of $\text{Isom}(\Sigma)$ into $\text{Diff}(\Sigma)$ is a homotopy equivalence for all spherical space forms Σ . The Hatcher conjecture generalizes the Smale conjecture [76] (proven by Hatcher in [42]), to which it reduces for $\Sigma = S^3$.

¹³Throughout we use ‘iff’ for ‘if and only if’.

2. 3-Manifolds

It is well known (e.g. [87]) that Einstein's equations (i.e. the constraints) pose no topological obstruction to Σ . Hence our Σ can be any closed 3-manifold. For simplicity (and no other reason) we shall exclude non-orientable manifolds and shall from now on simply say '3-manifold' if we mean *closed oriented* 3-manifold.

The main idea of understanding a general 3-manifold is to decompose it into simpler pieces by cutting it along embedded surfaces. Of most interest for us is the case where one cuts along 2-spheres, which results in the so-called *prime decomposition*. The inverse process, where two 3-manifolds are glued together by removing an embedded 3-disc in each of them and then identifying the remaining 2-sphere boundaries in an orientation reversing (with respect to their induced orientations) way is called *connected sum*. This is a well defined operation in the sense that the result is independent (up to homeomorphisms) of 1.) how the embedded 3-discs were chosen and 2.) what (orientation reversing) homeomorphism between 2-spheres is used for boundary identification (this is nicely discussed in § 10 of [7]). We write $\Sigma_1 \uplus \Sigma_2$ to denote the connected sum of Σ_1 with Σ_2 . The connected sum of a 3-manifold Σ with a 3-sphere, S^3 , is clearly homeomorphic to Σ .

Let us now introduce some important facts and notation. The classic source is [44], but we also wish to mention the beautiful presentation in [40]. Σ is called *prime* if $\Sigma = \Sigma_1 \uplus \Sigma_2$ implies that Σ_1 or Σ_2 is S^3 . Σ is called *irreducible* if every embedded 2-sphere bounds a 3-disc. Irreducibility implies primeness and the converse is almost true, the only exception being the *handle*, $S^1 \times S^2$, which is prime but clearly not irreducible (no $p \times S^2$ bounds a 3-disc). Irreducible prime manifolds have vanishing second fundamental group.¹⁴ The converse is true if every embedded 2-sphere that bounds a homotopy 3-disc also bounds a proper 3-disc; in other words, if fake 3-disks do not exist, which is equivalent to the Poincaré conjecture. So, if the Poincaré conjecture holds, a (closed orientable) 3-manifold P is prime iff either $\pi_2(P) = 0$ or $P = S^1 \times S^2$.

Many examples of irreducible 3-manifolds are provided by space forms, that is, manifolds which carry a metric of constant sectional curvature. These manifolds are covered by either S^3 or \mathbb{R}^3 and hence have trivial π_2 .

- Space forms of positive curvature (also called 'spherical space forms') are of the form S^3/G , where G is a finite freely acting subgroup of $SO(4)$. Next to the cyclic groups \mathbb{Z}_p these G e.g. include the $SU(2)$ double covers of the symmetry groups of n -prisms, the tetrahedron, the octahedron, and the icosahedron, as well as direct products of those with cyclic groups

¹⁴Note that this is *not* obvious from the definition of irreducibility, since non-zero elements of π_2 need not be representable as *embedded* 2-spheres. However, the *sphere theorem* for 3-manifolds (see Thm. 4.3 in [44]) implies that at least *some* non-zero element in π_2 can be so represented if π_2 is non-trivial.

of relatively prime (coprime) order.¹⁵ For example, the *lens spaces* $L(p, q)$, where q is coprime to p , are obtained by letting the generator of \mathbb{Z}_p act on $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ by $(z_1, z_2) \mapsto (rz_1, r^q z_2)$ with $r = \exp(2\pi i/p)$. See e.g. [86] for explicit presentations of the groups $G \subset SO(4)$.

- The flat space forms are of the form \mathbb{R}^3/G where \mathbb{R}^3 carries the Euclidean metric and G is a freely, properly-discontinuously acting subgroups of the group $E_3 = \mathbb{R}^3 \rtimes O(3)$ of Euclidean motions. There are six such groups leading to orientable compact quotients (see [88], Thm. 3.5.5 and Cor. 3.5.10): the lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$ of translations and five finite downward extensions¹⁶ of it by \mathbb{Z}_2 , \mathbb{Z}_3 , \mathbb{Z}_4 , \mathbb{Z}_6 , and $\mathbb{Z}_2 \times \mathbb{Z}_2$. These gives rise to the 3-torus T^3 and five spaces regularly covered by it.
- Space forms of negative curvature (also called ‘hyperbolic space forms’) are given by H^3/G , where $H^3 = \{(t, \vec{x}) \in \mathbb{R}^{(1,3)} : t = \sqrt{\vec{x}^2 + 1}\}$ is the hyperbola in Minkowski space and G is a freely, properly-discontinuously acting subgroup of the Lorentz group $O(1, 3)$ leading to orientable compact quotients. They are much harder to characterize explicitly.

Flat and hyperbolic space forms are covered by \mathbb{R}^3 so that all their homotopy groups higher than the first are trivial. The class of topological spaces for which $\pi_k = \{0\}$ for all $k > 1$ is generally called $K(\pi, 1)$ (Eilenberg MacLane spaces of type $(\pi, 1)$). In a sense, most primes are $K(\pi, 1)$ and much remains to be understood about them in general. Considerably more is known about a special subclass, the so called *sufficiently large* $K(\pi, 1)$ 3-manifolds or *Haken manifolds*. They are characterized by the property that they contain an embedded *incompressible* Riemann surface R_g , i.e. that if $e : R_g \rightarrow \Pi$ is the embedding then $e_* : \pi_1(R_g) \rightarrow \pi_1(\Pi)$ is injective.¹⁷ Simple examples are provided by the products $S^1 \times R_g$. An important conjecture in 3-manifold theory states that every irreducible 3-manifold with infinite fundamental group is virtually Haken, that is, finitely covered by a Haken manifold. If this is the case, any prime with infinite fundamental group allows for an immersion $e : R_g \rightarrow \Pi$ such that $e_* : \pi_1(R_g) \rightarrow \pi_1(\Pi)$ is injective.

¹⁵As far as I am aware, it is still an open conjecture that spherical space forms comprise all primes with finite fundamental group, even given the validity of the Poincaré conjecture. In other words, it is only conjectured that 3-manifolds covered by S^3 are of the form S^3/G where $G \subset SO(4)$ acting in the standard linear fashion. In [65] Milnor classified all finite groups that satisfied some necessary condition for having a free action on S^3 . The validity of the Smale conjecture [42] (which states that the embedding of $O(4)$ into $\text{Diff}(S^3)$ is a homotopy equivalence) eliminates those groups from the list which are not subgroups of $SO(4)$ [39]. What remains to be shown is that these groups do not admit inequivalent (equivalence being conjugation by some diffeomorphism) actions. The undecided cases are some cyclic groups of odd order; see [82].

¹⁶ Let G be a group with normal subgroup N and quotient $G/N = Q$. We call G either an *upward extension* of N by Q or a *downward extension* of Q by N ; see [9], p. XX.

¹⁷In other words, every loop on $e(R_g) \subset \Pi$ that bounds a 2-disc in Π (and is hence contractible in Π) also bounds a 2-disc on $i(R_g)$ (and is hence contractible in $e(R_g)$).

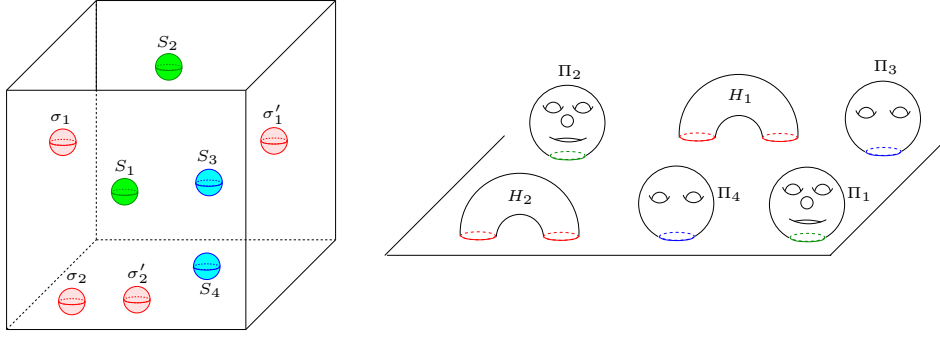


FIGURE 1. The connected sum of two handles $H_{1,2}$ and four irreducible primes Π_1, \dots, Π_4 , with Π_1 diffeomorphic to Π_2 and Π_3 diffeomorphic to Π_4 . For later application concerning the mapping class groups it is advisable to represent a handle as a cylinder $[0, 1] \times S^2$ with both ends, $0 \times S^2$ and $1 \times S^2$, separately attached by a connecting 2-sphere. The connecting spheres are denoted by σ_i, σ'_i for H_i ($i = 1, 2$) and S_i for Π_i ($i = 1, \dots, 4$). The left picture gives an *internal view*, in which only the connecting spheres are seen (and not what is behind), the right picture gives an *external view* from three dimensions onto a two-dimensional analogous situation that also reveals the topological structures behind the connecting spheres.

Now, given a connected sum ($N = n + m$)

$$\Sigma = \biguplus_{i=1}^N P_i = \left\{ \biguplus_{i=1}^n \Pi_i \right\} \uplus \left\{ \biguplus_{i=1}^m S^1 \times S^2 \right\}, \quad (2.1)$$

where notationally we distinguish between unspecified primes, denoted by P_i , and irreducible primes (i.e. those different from $S^1 \times S^2$), denoted by Π_i , so that $P_i = \Pi_i$ for $1 \leq i \leq n$ and $P_i = S^1 \times S^2$ for $n < i \leq N$. A simple application of van Kampen's rule gives that the fundamental group of the connected sum in 3 (and higher) dimensions is isomorphism to the free product (denoted by $*$) of the fundamental groups the factors (since the connecting spheres are simply connected):

$$\pi_1(\Sigma) = \pi_1(P_1) * \dots * \pi_1(P_N). \quad (2.2)$$

The converse is also true: a full decomposition of the group $\pi_1(\Sigma)$ into free products corresponds to a decomposition of Σ into the connected sum of primes (known as Kneser's conjecture).

The existence of prime decompositions was first shown by Kneser [54], the uniqueness (up to permutations of factors) by Milnor [66]. Regarding the latter we need to recall that we consider all manifolds to be oriented. Many orientable primes do not allow for orientation reversing self diffeomorphism; they are called

chiral. The table on p.922 of [24] lists which spherical and flat space forms are chiral; most of them are. Chiral primes with opposite orientation must therefore be considered as different prime manifolds.

Irreducible primes can be further decomposed by cutting them along 2-tori, which is the second major decomposition device in Thurston's *Geometrization Program* of 3-manifolds. Here we will not enter into this.

3. Mapping class groups

Mapping class groups can be studied through their action on the fundamental group. Consider the fundamental group of Σ based at $\infty \in \Sigma$. We write $\pi_1(\Sigma, \infty)$ or sometimes just π_1 for short. There are homomorphisms

$$h_F : \mathcal{G}_F(\Sigma) \rightarrow \text{Aut}(\pi_1), \quad (3.1)$$

$$h_\infty : \mathcal{G}_\infty(\Sigma) \rightarrow \text{Aut}(\pi_1), \quad (3.2)$$

$$h : \mathcal{G}(\Sigma) \rightarrow \text{Out}(\pi_1), \quad (3.3)$$

where the first two are given by

$$[\phi] \mapsto ([\gamma] \mapsto [\phi \circ \gamma]). \quad (3.4)$$

Here $[\phi]$ denotes the class of $\phi \in \text{Diff}_F(\Sigma)$ in $\text{Diff}_F(\Sigma)/\text{Diff}_F^0(\Sigma)$ (or of $\phi \in \text{Diff}_\infty(\Sigma)$ in $\text{Diff}_\infty(\Sigma)/\text{Diff}_\infty^0(\Sigma)$) and the other two square brackets the homotopy classes of the curves γ and $\phi \circ \gamma$. As regards (3.3), it is not difficult to see that in $\text{Diff}(\Sigma)$ any inner automorphism of $\pi_1(\Sigma, \infty)$ can be generated by a diffeomorphism that is connected to the identity (in $\text{Diff}(\Sigma)$, not in $\text{Diff}_\infty(\Sigma)$ or $\text{Diff}_F(\Sigma)$). Hence we have to factor out the inner automorphisms for the map h to account for the possibility to move the basepoint ∞ . More precise arguments are given in Section 3 of [29].

The images of h_∞ and h_F coincide but their domains may differ, i.e. the groups $\mathcal{G}_F(\Sigma)$ and $\mathcal{G}_\infty(\Sigma)$ are not necessarily isomorphic. Let us explain this a little further. Consider the fibration

$$\text{Diff}_F(\Sigma) \xrightarrow{i} \text{Diff}_\infty(\Sigma) \xrightarrow{p} \text{GL}^+(3, \mathbb{R}), \quad (3.5)$$

where $p(\phi) := \phi_*|_\infty$ (here we identify $\text{GL}^+(3, \mathbb{R})$ with the orientation preserving linear isomorphisms of $T_\infty(\Sigma)$). Associated with this fibration is a long exact sequence of homotopy groups, which ends with

$$1 \longrightarrow \pi_1(\text{Diff}_F(\Sigma)) \longrightarrow \pi_1(\text{Diff}_\infty(\Sigma)) \xrightarrow{p_*} \mathbb{Z}_2 \xrightarrow{\partial_*} \mathcal{G}_F(\Sigma) \xrightarrow{i_*} \mathcal{G}_\infty(\Sigma) \longrightarrow 1 \quad (3.6)$$

where the leftmost zero comes from $0 = \pi_2(\text{GL}^+(3, \mathbb{R}))$ and the \mathbb{Z}_2 in the middle is $\pi_1(\text{GL}^+(3, \mathbb{R}))$. Now, there are only two possibilities as regards the image of p_* :

1. $\text{Image}(p_*) = \mathbb{Z}_2 = \text{kernel}(\partial_*) \Rightarrow \mathcal{G}_F(\Sigma) \cong \mathcal{G}_\infty(\Sigma)$.
2. $\text{Image}(p_*) = \{0\} = \text{kernel}(\partial_*) \Rightarrow \mathcal{G}_F(\Sigma)$ is a downward extension (recall footnote 16) of $\mathcal{G}_\infty(\Sigma)$ by \mathbb{Z}_2 .

Let us focus on the second possibility. We first note that $\text{image}(\partial_*)$ lies in the kernel of h_F . This is true because the images of h_F and h_∞ coincide, so that $h_F \circ \partial_* = h_\infty \circ i_* \circ \partial_*$, which is the trivial map onto the identity in $\text{Aut}(\pi_1)$, since by exactness $i_* \circ \partial_*$ is the trivial map. The diffeomorphism that represents the non-trivial image of ∂_* can be represented by a rotation parallel to two concentric small spheres centered at ∞ ; see the left picture in Fig. 2. From this picture it also becomes clear that the diffeomorphism representing the 2π -rotation can be chosen of disjoint support from those diffeomorphisms representing other elements of $\mathcal{G}_F(\Sigma)$. Hence, if $\mathcal{G}_F(\Sigma)$ is a \mathbb{Z}_2 extension of $\mathcal{G}_\infty(\Sigma)$, it is *central*.

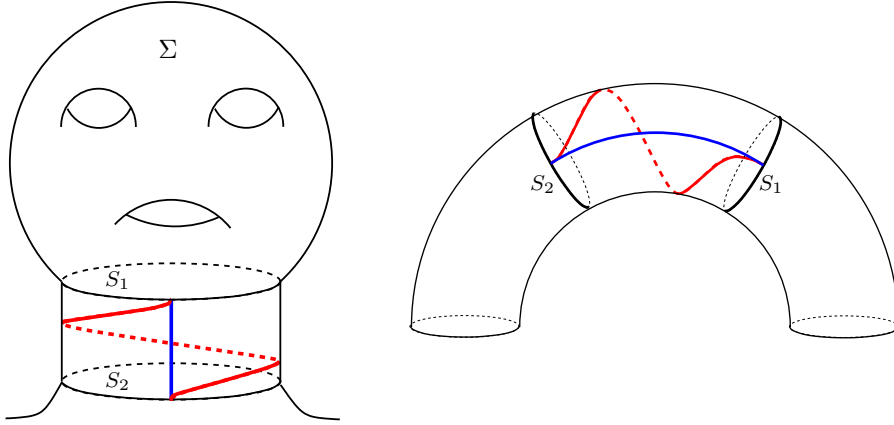


FIGURE 2. Both pictures show rotations parallel to spheres S_1 and S_2 . On the left, a rotation of the manifold Σ parallel to spheres both centered at ∞ , or, if $\Sigma = \Pi$ is a prime manifold in a connected sum, parallel to the connecting 2-sphere. On the right is a rotation parallel to two meridian spheres in a handle. The support of the diffeomorphism is on the cylinder bound by S_1 and S_2 . In either case its effect is depicted by the two curves connecting the two spheres: the diffeomorphism maps the straight to the curved line. This 2-dimensional representation is deceptive insofar as here the two straight and the curved lines are not homotopic (due to $\pi_1(S^1) = \mathbb{Z}$), whereas in 3 dimensions they are (due to the triviality of $\pi_1(S^2)$).

Manifolds for which $\mathcal{G}_F(\Sigma)$ is a (downward) central \mathbb{Z}_2 extension of $\mathcal{G}_\infty(\Sigma)$ are called *spinorial*. If they are used to model isolated systems (by removing ∞ and making the end asymptotically flat) their asymptotic symmetry group is not the Poincaré group (as discussed in [6]) but its double cover (i.e. the identity component of the homogeneous symmetry group is $SL(2, \mathbb{C})$ rather than the proper orthochronous Lorentz group). The origin of this is purely topological and has nothing to do with quantum theory, though the possible implications for quantum

gravity are particularly striking, as was first pointed out in a beautiful paper by Friedman and Sorkin [19]: it could open up the possibility to have half-integer spin states in pure gravity.¹⁸

A connected sum is spinorial iff it contains at least one spinorial prime [46]. Except for the lens spaces and handles all primes are spinorial, hence a 3-manifold is non-spinorial iff it is the connected sum of lens spaces and handles. Let us digress a little to explain this in somewhat more detail.

A small digression on spinoriality

That lens spaces and handles are non-spinorial is easily visualized. Just represent them in the usual fashion by embedding a lens or the cylinder $S^2 \times [0, 1]$ in \mathbb{R}^3 , with the boundary identifications understood. Place the base point, ∞ , on the vertical symmetry axis and observe that the rotation around this axis is compatible with the boundary identifications and therefore defines a diffeomorphism of the quotient space. A rotation parallel to two small spheres centered at ∞ can be continuously undone by rotating the body in \mathbb{R}^3 and keeping a neighborhood of ∞ fixed. This visualization also works for arbitrary connected sums of lens spaces and handles.

Spinoriality is much harder to prove. The following theorem has been shown by Hendriks ([46], Thm. 1 in § 4.3), and later in a more constructive fashion by Plotnick ([70], Thm. 7.4):

Theorem 3.1. *Let Σ be a closed (possibly non-orientable) 3-manifold and $\Sigma' := \Sigma - B_3$, where B_3 is an open 3-disc. A 2π -rotation in Σ' parallel to the boundary 2-sphere $\partial\Sigma'$ is homotopic to $\text{id}_{\Sigma'}$ rel. $\partial\Sigma'$ (i.e. fixing the boundary throughout) iff every prime summands of Σ is taken from the following list:*

- 1.) S_h^3/G , where S_h^3 is a homotopy sphere and G a finite freely acting group all Sylow subgroups of which are cyclic,
- 2.) the handle $S^1 \times S^2$,
- 3.) the (unique) non-orientable handle $S^1 \tilde{\times} S^2$,
- 4.) $S^1 \times \mathbb{RP}^3$, where \mathbb{RP}^3 denotes 3-dimensional real projective space.

Since here we excluded non-orientable manifolds from our discussion, we are not interested in 3.) and 4.). Clearly, S^3 is the only homotopy 3-sphere if the Poincaré conjecture holds. Of the remaining spherical space forms S^3/G the following have cyclic Sylow subgroups¹⁹:

- a.) $G = \mathbb{Z}_p$ (giving rise to the lens spaces),

¹⁸Topologically speaking, this is somewhat analogous to the similar mechanism in the Skyrme model [75], where loops in configuration space generated by 2π -rotations are non-contractible iff the skyrmion's winding number (its baryon number) is odd [23]). The analogy to the mechanism by which half-integer spin states can arise in gauge theories of integer spin fields [38, 33] is less close, as they need composite objects, e.g. from magnetic monopoles and electric charges.

¹⁹Each of the other groups contains as subgroup the 'quaternion group' $D_8^* := \{\pm 1, \pm i, \pm j, \pm k\}$, which is non abelian and of order $8 = 2^3$. Hence their 2-Sylow subgroups are not cyclic.

- b.) $G = D_{4m}^* \times \mathbb{Z}_p$ for $m = \text{odd}$ and $4m$ coprime to $p \geq 0$. Here D_{4m}^* is the $SU(2)$ double cover of $D_{2m} \subset SO(3)$, the order $2m$ symmetry group of the m -prism.
- c.) $G = D'_{2^k m} \times \mathbb{Z}_p$ for $m = \text{odd}$, $k > 3$, and $2^k m$ coprime to $p \geq 0$. Here $D'_{2^k m}$ is a (downward) central extension of D_{4m}^* by $\mathbb{Z}_{2^{k-2}}$.²⁰

Now there is a subtle point to be taken care of: that a diffeomorphism is *homotopic* to the identity means that there is a one-parameter family of *continuous* maps connecting it to the identity. This does not imply that it is *isotopic* to the identity, which means that there is a one-parameter family of *diffeomorphisms* connecting it to the identity. In case of the lens spaces it is easy to ‘see’ the isotopy, as briefly explained above. However, in the cases b.) and c.) it was proven by Friedman & Witt in [20] that the homotopy ensured by Thm. 3.1 does not generalize to an isotopy, so that these spaces again are spinorial. Taken together with Thm 3.1 this completes the proof of the statement that the only non-spinorial 3-manifolds are lens spaces, handles, and connected sums between them.

Note that this result also implies the existence of diffeomorphisms in $\text{Diff}(\Sigma)$ which are homotopic but not isotopic to the identity. For example, take the connected sum $\Sigma = \Pi_1 \uplus \Pi_2$ of two primes listed under b.) or c.). The 2π -rotation parallel to the connecting 2-sphere will now be an element of $\text{Diff}(\Sigma)$ that is homotopic but not isotopic to the identity [20]; see Fig. 3. This provides the first known example of such a behavior in 3-dimensions (in two dimensions it is known not to occur), though no example is known where this happens for a prime 3-manifold. In fact, that homotopy implies isotopy has been proven for a very large class of primes, including all spherical space forms, the handle $S^1 \times S^2$, Haken manifolds and many non-Haken $K(\pi, 1)$ (those which are Seifert fibered). See e.g. Thm. A1 of [26] for a list of references.

General Diffeomorphisms

It can be shown that if all primes in a connected sum satisfy the homotopy-implies-isotopy property, the kernel of h_F is isomorphic to $\mathbb{Z}_2^{m+n_s}$, where m is the number of handles and n_s the number of spinorial primes.²¹ This group is generated by the diffeomorphisms depicted in Fig. 2, one neck-twist (left picture) for each spinorial primes and one handle-twist (right picture) for each handle. It can also be shown that the mapping class groups of each prime injects into the mapping class groups of the connected sum in which it occurs [47]. This means that a diffeomorphism

²⁰We have $D_{2m} = \langle \alpha, \beta : \alpha\beta = \beta\alpha^{-1}, \alpha^m = \beta^2 = 1 \rangle$, where α is a $2\pi/m$ -rotation of the m -prism (vertical axis) and β is a π rotation about a horizontal axis. Then $D_{4m}^* = \langle \alpha, \beta : \alpha\beta = \beta\alpha^{-1}, \alpha^m = \beta^2 \rangle = \langle a, b : ab = ba^{-1}, a^m = b^4 = 1 \rangle$, where $a := \alpha\beta^2$ and $b := \beta$ (to show equivalence of these two presentations one needs that m is odd), and $D'_{2^k m} = \langle A, B : AB = BA^{-1}, A^m = B^{2^k} = 1 \rangle$. The center of $D'_{2^k m}$ is generated by B^2 and isomorphic to $\mathbb{Z}_{2^{k-1}}$. B^4 generates a central subgroup $\langle B^4 \rangle$ isomorphic to $\mathbb{Z}_{2^{k-2}}$ and $D'_{2^k m} / \langle B^4 \rangle \cong D_{4m}^*$.

²¹This follows from Thm. 1.5 in [62] together with the fact that for a manifold Π with vanishing π_2 two self-diffeomorphisms $\phi_{1,2}$ are homotopic if their associated maps $h_\infty : [\phi_{1,2}] \mapsto \text{Aut}(\pi_1(\Pi, \infty))$ coincide.

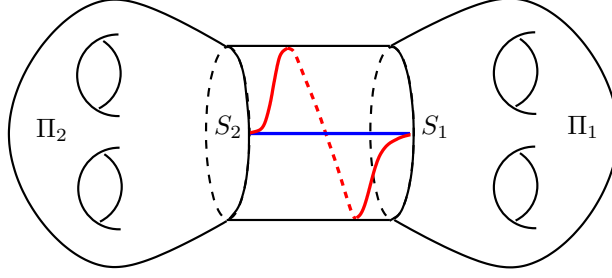


FIGURE 3. The connected sum of two irreducible primes Π_1 and Π_2 . The relative 2π -rotation is a diffeomorphism with support inside the cylinder bounded by the 2-spheres S_1 and S_2 . It transforms the straight line connecting S_1 and S_2 into the curved line. If $\Pi_i = S^3/G_i$ ($i = 1, 2$), where $G_{1,2} \in SO(4)$ are taken from the families $D_{4m}^* \times \mathbb{Z}_p$ or $D_{2^k m}' \times \mathbb{Z}_p$ mentioned under b.) and c.) in the text, this diffeomorphism is homotopic but not isotopic to the identity.

that has support in a prime factor P (we call such diffeomorphisms *internal*) and is not isotopic to the identity within in the space of all diffeomorphisms fixing the connecting 2-sphere is still not isotopic to the identity in $\text{Diff}_F(\Sigma)$. This statement would be false if $\text{Diff}_F(\Sigma)$ were replaced by $\text{Diff}(\Sigma)$. We will briefly come back to this point at the end of this section.

In analogy to (3.1), for each prime P , there is a map $h_F : \mathcal{G}_F(P) \rightarrow \text{Aut}(\pi_1(P))$ which is (almost) surjective in many cases. For example, if Π is Haken, h_F maps onto $\text{Aut}^+(\pi_1(\Pi))$ [41], the subgroup of orientation preserving automorphisms. If Π is not chiral, i.e. allows for orientation reversing self-diffeomorphisms, $\text{Aut}^+(\pi_1(\Pi)) \subset \text{Aut}(\pi_1(\Pi))$ is a subgroup of index two (hence normal). However, if Π is chiral, we have $\text{Aut}^+(\pi_1(\Pi)) = \text{Aut}(\pi_1(\Pi))$ and hence surjectivity. Since Haken manifolds are all spinorial, we can now say that their mapping class group is a central \mathbb{Z}_2 extension of $\text{Aut}^+(\pi_1(\Pi))$.

For spherical space forms the mapping class groups have all been determined in [86]. For $S^1 \times S^2$ it is $\mathbb{Z}_2 \times \mathbb{Z}_2$, where, say, the first \mathbb{Z}_2 is generated by the twist as depicted on the right in Fig. 2. The second \mathbb{Z}_2 corresponds to $\text{Aut}(\pi_1(\Pi)) = \text{Aut}(\mathbb{Z})$ and is generated by a reflection in the circle S^1 . If one thinks of a handle in a prime decomposition as a cylinder being attached with both ends, as depicted in Fig. 1, the latter diffeomorphism corresponds to exchanging the two cylinder ends (in an orientation preserving fashion), which is sometimes called a *spin* of a handle.

Suppose now that we are given a general connected sum (2.1) and that we know the mapping class group of each prime in terms of a finite presentation (finitely many generators and relations). We can then determine a finite presentation of $\mathcal{G}_F(\Sigma)$ by means of the so-called Foux-Rabinovitch presentation for the automorphism group of a free product of groups developed in [16, 17]; see also [63]

and [22]. Let generally

$$G = \underbrace{G_{(1)} * \cdots * G_{(n)}}_{\text{each } \not\cong \mathbb{Z}} * \underbrace{G_{(n+1)} * \cdots * G_{(n+m)}}_{\text{each } \cong \mathbb{Z}} \quad (3.7)$$

be a free product of groups corresponding to the decomposition (2.1). Let a set $\{g_{(i)1}, \dots, g_{(i)n_i}\}$ of generators for each $G_{(i)}$ be chosen. Clearly, for $n < i \leq n+m$ we have $n_i = 1$ so that we also write $g_{(i)1} =: g_{(i)}$. The generators of $\text{Aut}(G)$ can now be characterized by their action on these generators as follows (only non-trivial actions are listed):

- 1.) The generators of each $\text{Aut}(G_{(i)})$ for $1 \leq i \leq n$. As mapping-class generator these are called *internal*.
- 2.) The m generators σ_i ($1 \leq i \leq m$) whose effect is $\sigma_i(g_{(n+i)}) = g_{(n+i)}^{-1}$, i.e. generating $\text{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$. As mapping-class generator σ_i is called a *spin of the i -th handle*.
- 3.) One generator $\omega_{(i)(k)}$ for each pair of distinct but isomorphic groups $G_{(i)}, G_{(k)}$ ($1 \leq i, k \leq n+m$), whose effect it is to slotwise exchange the sets $\{g_{(i)1}, \dots, g_{(i)n_i}\}$ and $\{g_{(k)1}, \dots, g_{(k)n_k}\}$. (Here, for $1 \leq i, k \leq n$, we assume the generators of isomorphic groups to be chosen such that they correspond under a fiducial isomorphism, in particular $n_i = n_k$). As mapping-class generator $\omega_{(i)(k)}$ is called the *exchange of prime i with prime k* .
- 4.) One generator $\mu_{(i)j,(k)}$ for each $1 \leq i \leq n+m$, $1 \leq j \leq n_i$, and $1 \leq k \leq n$, whose effect is to map each generator $g_{(k)l}$ ($1 \leq l \leq n_k$) to $g_{(i)j}^{-1} \cdot g_{(k)l} \cdot g_{(i)j}$. As mapping-class generator $\mu_{(i)j,(k)}$ is called a *slide of the (irreducible) prime k through prime i along $g_{(i)j}$* .
- 5.) One pair of generators, $\lambda_{(i)j,(k)}$ and $\rho_{(i)j,(k)}$ for each $1 \leq i \leq n+m$, $1 \leq j \leq n_i$, and $n \leq k \leq n+m$. The effect of $\lambda_{(i)j,(k)}$ is to map each generator $g_{(k)}$ to $g_{(i)j}^{-1} \cdot g_{(k)}$ (i.e. left multiplication) and that of $\rho_{(i)j,(k)}$ to map each $g_{(k)}$ to $g_{(k)} \cdot g_{(i)j}$ (i.e. right multiplication). As mapping-class generator $\lambda_{(i)j,(k)}$ is called the *slide of the left end of handle k through prime i along $g_{(i)j}$* and $\rho_{(i)j,(k)}$ is called the *slide of the right end of handle k through prime i along $g_{(i)j}$* .

Of these generators the mapping class group realizes all those listed in 2.)-5.), but might leave out some in 1.) in case $h_F : \mathcal{G}_F(\Pi_i) \rightarrow \text{Aut}(\pi_1(\Pi_i))$ is not surjective for some $i \leq n$. In that case just replace 1. by the generators of the image of h_F (which might be a larger set than the generators of $\text{Aut}(\pi_1(\Pi_i))$). Finally, we have to add the generators of the kernel of h_F . As already stated, this kernel is given by the direct product of $n_s + m$ copies of \mathbb{Z}_2 , where n_s is the number of spinorial primes, if we assume the ‘homotopy-implies-isotopy’-property for all primes. In that case we have found all generators after adjoining these additional $n_s + m$ generators. A complete list of relations can then be found from the Foux-Rabinovitch relations

for $\text{Aut}(G) = G_1 * \cdots * G_{n+m}$ (see Chapter 5.1 of [63])²² and some added relations which the $n_s + m$ added generators have to satisfy. The latter are not difficult to find due to the simple geometric interpretation of the diffeomorphisms of Fig. 2 that represent the added generators.²³

The procedure just outlined reduces the problem of finding a presentation of $\mathcal{G}_F(\Sigma)$ to that of finding presentations $\mathcal{G}_F(\Pi_i)$ for each irreducible prime. As already stated, they are explicitly known for all spherical space forms. We also mentioned that for Haken manifolds $\mathcal{G}_F(\Pi_i)$ is a \mathbb{Z}_2 extension of $\text{Aut}^+(\pi_1)$, which in simple cases allows to find an explicit presentation. For example, for the 3-torus we have $\text{Aut}^+(\pi_1 = \mathbb{Z}^3) = SL(3, \mathbb{Z})$ and the appropriate central \mathbb{Z}_2 extension can be shown to be given by the Steinberg group $St(3, \mathbb{Z})$, a presentation of which may e.g. be found in § 10 of [67].

All the generators listed in 2.-5. can be realized by appropriate diffeomorphisms. This is not difficult to see for 2. and 3., as diffeomorphisms that ‘spin’ a handle or ‘exchange’ two diffeomorphic primes are easily visualized. A visualization of the slide transformations 4. and 5. is attempted in Fig. 4: The general idea—and this is where the name ‘slide’ derives from—is similar to that of rotation of parallel (i.e. concentric) spheres explained in Fig. 2. But now we take two ‘parallel’ (i.e. coaxial) tori, T_1 and T_2 , and consider a diffeomorphism whose support is confined to the region between them. This toroidal region is of topology $[1, 2] \times S^1 \times S^1$ (i.e. does not contain prime summands) and hence is foliated by a one parameter (r) family of parallel (coaxial) tori $T_r = r \times S^1 \times S^1$, where $r \in [1, 2]$. Each of these tori we think of as being coordinatized in the standard fashion by two angles, θ and φ with range $[0, 2\pi]$ each, where θ labels the latitude and φ the longitude (the circles of constant φ are the small ones that become contractible in the solid torus). The slide now corresponds to a diffeomorphism which is the identity outside the toroidal region and which inside ($1 \leq r \leq 2$) is given by

$$(r, \theta, \varphi) \mapsto (r, \theta, \varphi + \beta(2 - r)), \quad (3.8)$$

where β is a C^∞ step-function $\beta : [0, 1] \rightarrow [0, 2\pi]$ with $\beta(0) = 0$ and $\beta(1) = 2\pi$. In Fig. 4, the loop γ generates a non-trivial element $[\gamma] \in \pi_1(\Pi, p)$ (the non-triviality is indicated by the little knot inside Π for lack of better representation). This loop γ , after having been acted on by the slide, will first follow ℓ and go through the handle, then travel through Π as before, and finally travel the handle in a reversed sense. That is, the slide conjugates $[\gamma]$ with $[\ell]$.

²²The original papers by Fouxé-Rabinovitch [17, 16] contained some errors in the relations which were corrected in [63]; see also [22].

²³The added generators are internal transformations (i.e. have support within the prime factors) and hence behave naturally under exchanges. They commute with all other internal diffeomorphisms and slides of the prime in question, since their supports may be chosen to be disjoint from the diffeomorphisms representing the other internal diffeomorphisms and slides. They also commute with slides of other primes *through* the one in question, since their action on such slides (by conjugation) is a slide along a curve isotopic to the original one, which defines the same mapping class. Finally, the conjugation of a handle’s ‘twist’ (right picture in Fig. 2) with a spin of that handle is isotopic to the original twist.

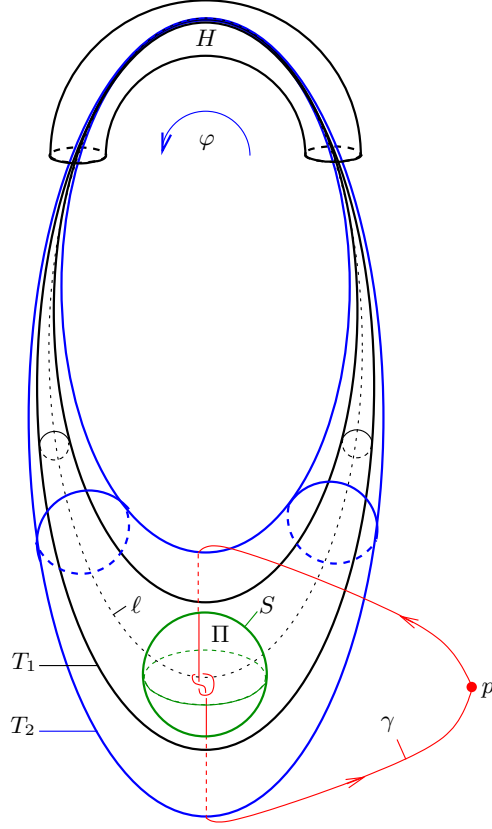


FIGURE 4. The slide of a prime as seen from the ‘inside view’ (compare the left picture in Fig.1) through a handle H in the background. The prime to be slid, Π , hides behind its connecting 2-sphere S . The loop ℓ , representing the generator of $\pi_1(H)$ along which the prime is slid, is thickened to two coaxial tori, T_1 and T_2 , such that the prime is contained in the inner torus T_1 . The diffeomorphism in the toroidal region is given by (3.8), where the angle φ measures the axial direction. The slide acts on $[\gamma] \in \pi_1(\Pi, p) \subset \pi_1(\Sigma, p)$ (the non-triviality of which being indicated by the little knot inside Π) by conjugation with $[\ell]$, when the latter is appropriately considered as element of $\pi_1(\Sigma, p)$.

The slides of irreducible primes described in 4.), or the slides of ends of handles described in 5.), are then obtained by choosing the tori such that the only connecting sphere contained inside the inner torus is that of the prime, or handle-end, to be slid. The common axis of the tori trace out a non-contractible

loop through another prime, the prime through which the first one is slid. All this is depicted in Fig. 4.

Of crucial importance is the different behavior of slides in 4.) on one hand, and slides in 5.) on the other. Algebraically this has to do with the different behavior, as regards the automorphism group of the free product, of the free factors \mathbb{Z} on one hand and the non-free factors $G_{(i)}$ on the other. Whereas left or right multiplication of all elements in one \mathbb{Z} factor with any element from the complementary free product defines an automorphism of G , this is not true for the non-free factors. Here only conjugation defines an automorphism. Geometrically this means that we have to consider slides of both ends of a handle separately in order to be able to generate the automorphism group. It is for this reason that we pictured the handles in Fig. 1 as being attached to the base manifold with two rather than just one connecting sphere.

In passing we remark that certain important interior diffeomorphism (i.e. falling under case 1.) above) have an interpretation in terms of certain *internal* slides. Imagine Fig. 4 as an *inside* view from some prime Π , i.e. everything seen on Fig. 4 is inside Π . The handle H , too, is now interpreted as some topological structure of Π itself that gives rise to non-contractible loops within Π . S is again the connecting sphere of Π , now seen from the inside, beyond which the part of the manifold Σ outside Π lies. The diffeomorphism represented by Fig. 4 then slides the connecting sphere S once around the loop ℓ in Π . Its effect is to conjugate each element of $\pi_1(\Pi)$ by $[\ell] \in \pi_1(\Pi)$. Hence we see that such internal slides generate all *internal* automorphisms for each irreducible factor Π . In case of handles, there are no non-trivial inner automorphisms, and the only non-trivial outer automorphism ($\mathbb{Z} \mapsto -\mathbb{Z}$) is realized by spinning the handle, as already mentioned.

Having said that, we will from now on always understand by ‘slides’ *external* transformations as depicted in 4, unless explicitly stated otherwise (cf. last remark at the end of this section). But let us for the moment forget about slides altogether and focus attention only on those mapping classes listed in 1.)-3.), i.e. internal transformations and exchanges. In doing this we think of a spin of a handle as internal, which we may do as long as no slides are considered. It is tempting to think of the manifold Σ as being composed of $N = n + m$ ‘particles’ from d species, each with its own characteristic internal symmetry group G_r , $1 \leq r \leq d$. In this analogy diffeomorphic primes correspond to particles of one species and the symmetry groups G_r correspond to $\mathcal{G}_F(P_r)$. Let there be n_r primes in the r -th diffeomorphism class, so that $\sum_{r=1}^d n_r = N$. In this ‘particle picture’ the symmetry group would be a semi-direct product of the internal symmetry group, G^I , with an external symmetry group, G^E , both respectively given by

$$G^I := \prod_{i=1}^{n_1} G_1 \times \cdots \times \prod_{i=1}^{n_d} G_d, \quad (3.9a)$$

$$G^E := S_{n_1} \times \cdots \times S_{n_d}, \quad (3.9b)$$

where here S_n denotes the order $n!$ permutation group of n objects. The semi-direct product is characterized through the homomorphism $\theta : G^E \rightarrow \text{Aut}(G^I)$,

where $\theta = \theta_1 \times \cdots \times \theta_d$ and

$$\begin{aligned} \theta_i : S_{n_i} &\rightarrow \text{Aut} \left(\prod^{n_i} G_i \right) \\ \sigma &\mapsto \theta_i(\sigma) : (g_1, \dots, g_{n_i}) \mapsto (g_{\sigma(1)}, \dots, g_{\sigma(n_i)}). \end{aligned} \quad (3.10)$$

The semi-direct product $G^I \rtimes G^E$ with respect to θ is now defined by the following multiplication law: let $\gamma_i \in \prod^{n_i} G_i$, $i = 1, \dots, d$, then

$$\begin{aligned} &(\gamma'_1, \dots, \gamma'_d; \sigma'_1, \dots, \sigma'_d)(\gamma_1, \dots, \gamma_d; \sigma_1, \dots, \sigma_d) \\ &= (\gamma'_1 \theta_1(\sigma'_1)(\gamma_1), \dots, \gamma'_d \theta_d(\sigma'_d)(\gamma_d); \sigma'_1 \sigma_1, \dots, \sigma'_d \sigma_d). \end{aligned} \quad (3.11)$$

We call $G^P = G^I \rtimes G^E$ the *particle group*. From the discussion above it is clear that this group is a subgroup of the mapping class group. But we also had to consider slides which were neither internal nor exchange diffeomorphisms and which are not compatible with this simple particle picture, since they mix internal and external points of the manifold. How much do the slides upset the particle picture? For example, consider the normal closure, G^S , of slides (i.e. the smallest normal subgroup in the group of mapping classes that contains all slides). Does it have a non-trivial intersection with G^P , i.e. is $G^P \cap G^S \neq \{1\}$? If this is the case $\mathcal{G}_F(\Sigma)/G^S$ will be a non-trivial factor of G^P . Representations whose kernels contain G^S will then not be able to display all particle symmetries. This would only be the case if $G^P \cap G^S = \{1\}$.

Questions of this type have been addressed and partly answered in [29] (see also [80]). Here are some typical results:

Proposition 3.2. $G^P \cap G^S = \{1\}$ and $\mathcal{G}_F(\Sigma) = G^S \rtimes G^P$ if Σ contains no handle in its prime decomposition.

Proposition 3.3. G^S is perfect if Σ contains at least 3 handles in its prime decomposition.

The last proposition implies that slides cannot be seen in abelian representations of mapping class groups of manifolds with at least three handles. In [29] an explicit presentation with four generators of the mapping class group of the connected sum of $n \geq 3$ handles was given and the following result was shown:

Proposition 3.4. Let $\Sigma = \bigoplus^n S^1 \times S^2$ where $n \geq 3$. Then $\mathcal{G}_F(\Sigma)/G^S \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ where one \mathbb{Z}_2 is generated by the twist (right picture in Fig. 3) of, say, the first handle and the other \mathbb{Z}_2 by either the exchange of, say, the first and second handle or the spin of, say, the first handle. Hence we have a strict correlation between spins and exchanges of handles. Generally, given a representation ρ of $\mathcal{G}_F(\Sigma)$, the following statements are equivalent:

- a.) ρ is abelian,
- b.) slides are in the kernel of ρ ,
- c.) ρ strictly correlates exchanges and spins (i.e. $\rho(\text{spin}) = \rho(\text{exchange})$),
- d.) slides and exchanges commute under ρ .

Reference [29] also deals with connected sums of arbitrarily many real projective spaces \mathbb{RP}^3 . A presentation in terms of three generators was written down and various features studied. Since projective spaces are the most simple lens spaces ($\mathbb{RP}^3 = L(2,1)$) they are not spinorial, as one easily visualizes. Since the automorphism group of $\pi_1(\mathbb{RP}^3) = \mathbb{Z}_2$ is trivial there are no non-trivial mapping classes from internal diffeomorphisms (\mathbb{RP}^3 satisfies the homotopy-implies-isotopy property). Therefore, the particle group G^P is just the permutation group S_n and the mapping class group is the semi-direct product $G^S \rtimes S_n$, according to Prop. 3.2. A very interesting systematic study of representations of this group was started in [80] using Mackey theory (theory of induced representations). For illustrative purposes we consider in some detail the case of the connected sum of just two projective spaces in the next section, also considered in [3] and [80].

Even though we here restrict attention to frame-fixing diffeomorphisms, which is physically well motivated, we nevertheless wish to end this section with a few remarks that give an idea of the essential changes that result if we relaxed from $\text{Diff}_F(\Sigma)$ (or $\text{Diff}_\infty(\Sigma)$) to $\text{Diff}(\Sigma)$. In the frame-fixing context, non-trivial mapping classes of prime factors (i.e. generated by internal diffeomorphisms) are non-trivial mapping classes in the total manifold Σ . In other words, there is an injection $\mathcal{G}_F(P) \rightarrow \mathcal{G}_F(\Sigma)$ [47]. As already remarked above, this is not true if $\mathcal{G}_F(\Sigma)$ is replaced by $\mathcal{G}(\Sigma)$. In fact, it follows from the above that any diffeomorphism in $\text{Diff}_F(\Sigma)$ whose image under h_F in $\text{Aut}(\pi_1(\Sigma))$ is an *inner* automorphism is isotopic in $\text{Diff}(\Sigma)$ to transformations of the type depicted in Fig. 2. Hence there are generally many inner diffeomorphisms of prime factors P which represent non-trivial elements of $\text{Diff}_F(P)$ but trivial elements in $\text{Diff}(\Sigma)$. Also, the distinction between inner and non-inner (exchanges and slides) diffeomorphisms ceases to be meaningful in $\text{Diff}(\Sigma)$. A trivial example is the diffeomorphism depicted in Fig. 3, i.e. the 2π -rotation parallel to the common connecting sphere of two irreducible prime manifolds. Up to isotopy in $\text{Diff}(\Sigma)$ it can clearly be considered as inner diffeomorphism of *either* prime. A less trivial example is the following: consider $\Sigma = \Pi_1 \uplus \Pi_2$, where $\{g_{(1)1}, \dots, g_{(1)n_1}\}$ and $\{g_{(2)1}, \dots, g_{(2)n_2}\}$ are the generators of $\pi_1(\Pi_1)$ and $\pi_1(\Pi_2)$ respectively. As explained above, a slide of Π_1 through Π_2 along, say, $g_{(2)1}^{-1}$ acts on these sets of generators by conjugating each in the first set with $g_{(2)1}^{-1}$. On the other hand, each inner automorphism of $\pi_1(\Sigma)$ can be produced by a diffeomorphism that is isotopic to the identity in $\text{Diff}(\Sigma)$.²⁴ Taking for that the diffeomorphism that conjugates $\pi_1(\Sigma)$ by $g_{(2)1}$ we see that the slide just considered is isotopic (in $\text{Diff}(\Sigma)$, not in $\text{Diff}_F(\Sigma)$ or $\text{Diff}_\infty(\Sigma)$) to a diffeomorphism that leaves the $g_{(1)i}$ untouched and conjugates the $g_{(2)i}$ by $g_{(2)1}$. This, in turn, can

²⁴That there is a diffeomorphism generating any inner automorphism of $\pi_1(\Sigma)$ is shown exactly as in our discussion of ‘internal slides’ above. See Fig. 4, where now everything is in Σ and the inside of T_1 is taken to be a solid torus, i.e. there is *no* prime factor Π inside T_1 . Then it is obvious that the slide depicted is isotopic to the identity in $\text{Diff}(\Sigma)$ by a diffeomorphism whose support is inside T_2 but extends into the inside of T_1 : just take the isotopy $[0,1] \ni s \mapsto \phi_s : (r, \theta, \varphi) \mapsto (r, \theta, s\beta(2 - r))$, where $\beta : [0,1] \rightarrow [0,2\pi]$ is the step function as in (3.8), continued by the constant value 2π to $[1,2]$.

be represented by an inner diffeomorphism of Π_2 (an internal slide, possibly with a 2π -rotation). This shows that the original slide of Π_1 through Π_2 is isotopic in $\text{Diff}(\Sigma)$ (not in $\text{Diff}_F(\Sigma)$ or $\text{Diff}_\infty(\Sigma)$) to an internal diffeomorphism of Π_2 .²⁵ For example, if Π_2 is a lens space, which is non-spinorial and has abelian fundamental group, the internal diffeomorphism just considered is clearly isotopic to the identity. Hence the original slide of the first prime through the lens space is isotopic to the identity in $\text{Diff}(\Sigma)$, but not in $\text{Diff}_\infty(\Sigma)$ or $\text{Diff}_F(\Sigma)$. All this shows that the division of mapping-class generators into internal diffeomorphisms, exchanges, and slides only makes sense in $\text{Diff}_\infty(\Sigma)$ and $\text{Diff}_F(\Sigma)$, but not in $\text{Diff}(\Sigma)$.

4. A simple yet non-trivial example

In this section we wish to discuss in detail the mapping class group $\mathcal{G}_F(\Sigma)$ for $\Sigma = \mathbb{RP}^3 \uplus \mathbb{RP}^3$. Before doing this, let us say a few words on how the single \mathbb{RP}^3 manifold can arise in an exact black-hole solution in General Relativity.

4.1. The \mathbb{RP}^3 geon

Recall that we limited attention to asymptotically flat manifolds with a single end (no ‘internal infinity’). Is this not too severe a restriction? After all, we know that the (maximally extended) manifold with one (uncharged, non-rotating) black hole is the Kruskal manifold [55] (see also Chapter 5.5. in [43]), in which space has *two* ends. Figure 5 shows the familiar conformal diagram, where the asymptotically flat ends lie in regions I and III. In Kruskal coordinates²⁶ (T, X, θ, φ) , where T and X each range in $(-\infty, \infty)$ obeying $T^2 - X^2 < 1$, the Kruskal metric reads (as usual, we write $d\Omega^2$ for $d\theta^2 + \sin^2 \theta d\varphi^2$):

$$g = \frac{32m^2}{r} \exp(-r/2m) (-dT^2 + dX^2) + r^2 d\Omega^2, \quad (4.1)$$

where r is a function of T and X , implicitly defined by

$$((r/2m) - 1) \exp(r/2m) = X^2 - T^2 \quad (4.2)$$

and where $m > 0$ represents the mass of the hole in geometric units. The metric is spherically symmetric and allows for the additional Killing field²⁷

$$K = \frac{1}{4m} (X \partial_T + T \partial_X), \quad (4.3)$$

which is timelike for $|X| > |T|$ and spacelike for $|X| < |T|$.

²⁵This point is not correctly taken care of in [12], where it is argued that a slide of one prime through the other is *always* either isotopic to the identity or the relative 2π -rotation of both primes ([12], p.1162). But this is only true if the loop along which is slid generates a central element of $\pi_1(\Pi_2)$. In particular, the isotopy claimed in the last sentence of the footnote on p.1162 of [12] cannot exist in general. However, the application in [12] is eventually restricted to primes with abelian fundamental group so that this difficulty does not affect the specific conclusions drawn in [12]. I thank Fay Dowker and Bob Gompf for discussions of that point.

²⁶Kruskal [55] uses (v, u) Hawking Ellis [43] (t', x') for what we call (T, X) .

²⁷That K is Killing is immediate, since r depends only on the combination $X^2 - T^2$ which is annihilated by K .

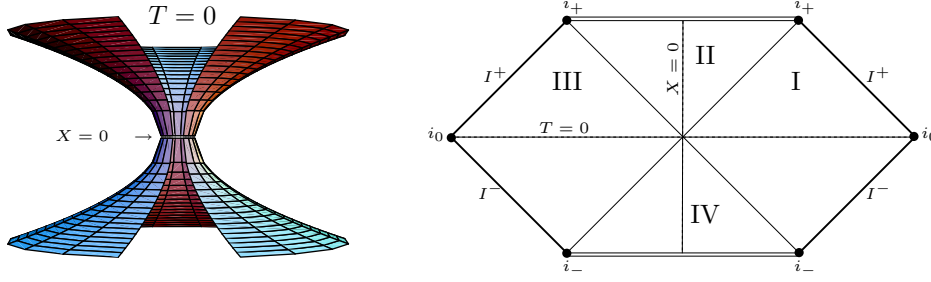


FIGURE 5. To the right is the conformal (Penrose) diagram of Kruskal spacetime in which each point of this 2-dimensional representation corresponds to a 2-sphere (an orbit of the symmetry group of spatial rotations). The asymptotic regions are i_0 (spacelike infinity), I^\pm (future/past lightlike infinity), and i^\pm (future/past timelike infinity). The diamond and triangular shaped regions I and II correspond to the exterior ($r > 2m$) and interior ($0 < r < 2m$) Schwarzschild spacetime respectively, the interior being the black hole. The triangular region IV is the time reverse of II, a white hole. Region III is another asymptotically flat end isometric to the exterior Schwarzschild region I. The double horizontal lines on top and bottom represent the singularities ($r = 0$) of the black and white hole respectively. The left picture shows an embedding diagram of the hypersurface $T = 0$ (central horizontal line in the conformal diagram) that serves to visualize its geometry. Its minimal 2-sphere at the throat corresponds to the intersection of the hyperplanes $T = 0$ and $X = 0$ (bifurcate Killing Horizon).

The familiar exterior Schwarzschild solution is given by region I in Fig. 5, where the transformation from Schwarzschild coordinates (t, r, θ, φ) , where $2m < r < \infty$ and $-\infty < t < \infty$, to Kruskal coordinates is given by

$$T = \sqrt{(r/2m) - 1} \exp(r/4m) \sinh(t/4m), \quad (4.4a)$$

$$X = \sqrt{(r/2m) - 1} \exp(r/4m) \cosh(t/4m). \quad (4.4b)$$

This obviously just covers region I: $X > |T|$. In Schwarzschild coordinates the Killing field (4.3) just becomes $K = \partial_t$.

Now consider the following discrete isometry of the Kruskal manifold:

$$J : (T, X, \theta, \varphi) \mapsto (T, -X, \pi - \theta, \varphi + \pi). \quad (4.5)$$

It generates a freely acting group \mathbb{Z}_2 of smooth isometries which preserve space- as well as time-orientation. Hence the quotient is a smooth space- and time-orientable

manifold, the \mathbb{RP}^3 *geon*.²⁸ Its conformal diagram is just given by cutting away the $X < 0$ part (everything to the left of the vertical $X = 0$ line) in Fig. 5 and taking into account that each point on the remaining edge, $X = 0$, now corresponds to a 2-sphere with antipodal identification, i.e. a \mathbb{RP}^2 which is non-orientable. The spacelike hypersurface $T = 0$ has now the topology of the once punctured \mathbb{RP}^3 . In the left picture of Fig. 5 this corresponds to cutting away the lower half and eliminating the inner boundary 2-sphere $X = 0$ by identifying antipodal points. The latter then becomes a minimal one-sided non-orientable surface in the orientable space-section of topology $\mathbb{RP}^3 - \{\text{point}\}$. The \mathbb{RP}^3 geon isometrically contains the exterior Schwarzschild spacetime (region I) with timelike Killing field K . But K ceases to exist globally on the geon spacetime since it reverses direction under (4.5).

Even though the Kruskal spacetime and its quotient are, geometrically speaking, locally indistinguishable, their physical properties are different. In particular, the thermodynamic properties of quantum fields are different. For details we refer to [56] and references therein. We also remark that the mapping-class group $\mathcal{G}_F(\mathbb{RP}^3)$ is trivial [86], as are the higher homotopy groups of $\mathcal{S}_F(\mathbb{RP}^3)$ [26]. Equation (1.16) then shows that the configuration space $\mathcal{S}_F(\mathbb{RP}^3)$ is (weakly) homotopically contractible. In fact, the three-sphere S^3 and the real projective 3-space \mathbb{RP}^3 are the only 3-manifolds for which this is true; see [24] (table on p. 922).

4.2. The connected sum $\mathbb{RP}^3 \uplus \mathbb{RP}^3$

Asymptotically flat initial data on the once punctured manifold $\mathbb{RP}^3 \uplus \mathbb{RP}^3 - \{\text{point}\}$ can be explicitly constructed. For this one considers time-symmetric conformally-flat initial data. The constraints (1.4) then simply reduce to the Laplace equation for a positive function Φ , where Φ^4 is the conformal factor. The ‘method of images’ known from electrostatics can then be employed to construct special solutions with reflection symmetries about two 2-spheres. The topology of the initial data surface is that of \mathbb{R}^3 with two disjoint open 3-discs excised. This excision leaves two inner boundaries of S^2 topology on each of which antipodal points are identified. The metric is constructed in such a way that it projects in a smooth fashion to the resulting quotient manifold whose topology is that of $\mathbb{RP}^3 \uplus \mathbb{RP}^3 - \{\text{point}\}$. Details and analytic expressions are given in [30]. These data describe two black holes momentarily at rest. The spacetime they involve into (via (1.3)) is not known analytically, but since the analytical form of the data is very similar indeed to the form of the Misner-wormhole data (cf. [30]), which were often employed in numerical studies, I would not expect the numerical evolution

²⁸The \mathbb{RP}^3 geon is different from the two mutually different ‘elliptic interpretations’ of the Kruskal spacetime discussed in the literature by Rindler, Gibbons, and others. In [72] the identification map considered is $J' : (T, X, \theta, \varphi) \mapsto (-T, -X, \theta, \varphi)$, which gives rise to singularities on the set of fixed-points (a two-sphere) $T = X = 0$. Gibbons [21] takes $J'' : (T, X, \theta, \varphi) \mapsto (-T, -X, \pi - \theta, \varphi + \pi)$, which is fixed-point free, preserves the Killing field (4.3) (which our map J does not), but does not preserve time-orientation. J'' was already considered in 1957 by Misner & Wheeler (Section 4.2 in [68]), albeit in so-called ‘isotropic Schwarzschild coordinates’, which only cover the exterior regions I and III of the Kruskal manifold.

to pose any additional difficulties. All this is just to show that the once punctured manifold $\mathbb{RP}^3 \uplus \mathbb{RP}^3$ is not as far fetched in General Relativity as it might seem at first: it is as good, and no more complicated than, the Misner wormhole which is the standard black-hole data set in numerical studies of head-on collisions of equal-mass black holes.²⁹

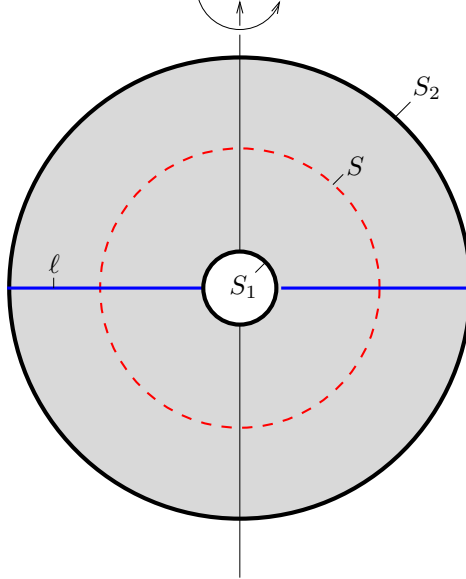


FIGURE 6. Visualization of $\mathbb{RP}^3 \uplus \mathbb{RP}^3$, which corresponds to the shaded region between the 2-spheres S_1 and S_2 , on each of which antipodal points are identified. As indicated, the whole picture is to be thought of as rotating about the vertical axis, except for the solid vertical line ℓ , which, like any other radial line, corresponds to a closed loop, showing that $\mathbb{RP}^3 \uplus \mathbb{RP}^3$ is fibered by circles over \mathbb{RP}^2 . Each fiber intersects the connecting 2-sphere, S , in two distinct points.

We wish to study and visualize the mapping class group $\mathcal{G}_F(\Sigma)$ where $\Sigma = \mathbb{RP}^3 \uplus \mathbb{RP}^3$. We represent Σ by the annular region depicted in Fig. 6, which one should think of as representing a 3-dimensional spherical shell inbetween the outer boundary 2-sphere S_2 and the inner boundary 2-sphere S_1 . In addition, on each boundary 2-sphere we identify antipodal points. The result is the connected sum $\mathbb{RP}^3 \uplus \mathbb{RP}^3$ where we might take S for the connecting 2-sphere that lies ‘half way’

²⁹The \mathbb{RP}^3 data even have certain advantages: they generalize to data where the masses of the black holes are not equal (for the wormhole identification the masses need to be equal) and even to data for any number of holes with arbitrary masses (in which case the holes may not be ‘too close’).

between S_1 and S_2 in Fig. 6. The radial lines, like ℓ , fiber the space in loops showing that $\mathbb{RP}^3 \uplus \mathbb{RP}^3$ is an S^1 bundle over \mathbb{RP}^2 .³⁰ Interestingly, it is doubly covered by the prime manifold $S^1 \times S^2$, the corresponding deck transformation of the latter being $(\psi, \theta, \varphi) \mapsto (2\pi - \psi, \pi - \theta, \varphi + \pi)$, where $\psi \in [0, 2\pi]$ coordinatizes S^1 and (θ, φ) are the standard spherical polar coordinates on S^2 . Note that this deck transformation does not commute with the $SO(2)$ part of the obvious transitive $SO(2) \times SO(3)$ action on $S^1 \times S^2$, so that only a residual $SO(3)$ action remains on $\mathbb{RP}^3 \uplus \mathbb{RP}^3$ whose orbits are 2-spheres except for two \mathbb{RP}^2 s.³¹ It is known to be the only example of a (closed orientable) 3-manifold that is a proper connected sum and covered by a prime.³²

The fundamental group is the free product of two \mathbb{Z}_2 :

$$\pi_1(\mathbb{RP}^3 \uplus \mathbb{RP}^3) \cong \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b : a^2 = b^2 = 1 \rangle. \quad (4.6)$$

Two loops representing the generators a and b are shown in Fig. 7. Their product,

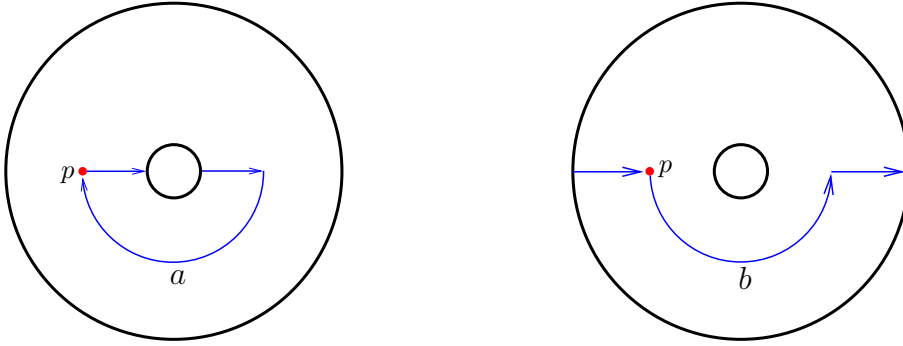


FIGURE 7. Loops a (left picture) and b (right picture) in $\mathbb{RP}^3 \uplus \mathbb{RP}^3$ that generate the fundamental group $\mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b : a^2 = b^2 = 1 \rangle$ based at point p . The product loop, $c := ab$ (first a then b), is homotopic to the loop ℓ in Fig. 6, that is, to a circle fiber.

$ab =: c$, is homotopic to a circle fiber. Replacing the generator b by ac (recall $a^2 = 1$), the presentation (4.6) can now be written in terms of a and c :

$$\pi_1(\mathbb{RP}^3 \uplus \mathbb{RP}^3) = \langle a, c : a^2 = 1, aca^{-1} = c^{-1} \rangle \cong \mathbb{Z} \rtimes \mathbb{Z}_2. \quad (4.7)$$

where \mathbb{Z}_2 (generated by a) acts as the automorphism $c \mapsto c^{-1}$ on the generator c of \mathbb{Z} . This corresponds to the structure of $\mathbb{RP}^3 \uplus \mathbb{RP}^3$ as S^1 -fiber bundle over \mathbb{RP}^2

³⁰ $\mathbb{RP}^3 \uplus \mathbb{RP}^3$ is an example of a Seifert fibered space without exceptional fibers, that was already mentioned explicitly in Seifert's thesis [74].

³¹ \mathbb{RP}^3 is trivially homogeneous, being $SO(3)$. The punctured space $\mathbb{RP}^3 - \{\text{point}\}$ is also homogeneous, since it may be identified with the space of all hyperplanes (not necessarily through the origin) in \mathbb{R}^3 , on which the group $E_3 = \mathbb{R}^3 \rtimes SO(3)$ of Euclidean motions clearly acts transitively with stabilizers isomorphic to E_2 ; hence $\mathbb{RP}^3 - \{\text{point}\} \cong E_3/E_2$.

³²Clearly, no proper connected sum ($\pi_2 \neq 0$) can be covered by an irreducible prime ($\pi_2 = 0$).

with π_1 (base) acting on π_1 (fibre). Algebraically, the normal subgroup \mathbb{Z} generated by c is just the subgroup of words in a, b containing an even number of letters.

The generators of mapping classes are the (unique) exchange, ω , the slide μ_{12} of prime 2 through prime 1 (there is only one generator of π_1 for each prime and hence a unique generating slide through each prime), and the slide μ_{21} of prime 1 through prime 2. The relations between them are $\omega^2 = \mu_{12}^2 = \mu_{21}^2 = 1$ and $\omega\mu_{12}\omega^{-1} = \mu_{21}$. There are no other relations, as one may explicitly check using the Fouxé-Rabinovitch relations [63]. The particle group G^P is just \mathbb{Z}_2 , generated by ω , and the slide subgroup G^S is $\mathbb{Z}_2 * \mathbb{Z}_2$, generated by μ_{12} and μ_{21} . We have $\mathcal{G}_F(\Sigma) = G^S \rtimes G^P = (\mathbb{Z}_2 * \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, where \mathbb{Z}_2 acts on $\mathbb{Z}_2 * \mathbb{Z}_2$ by permuting the factors.

Instead, we may get a presentation in terms of just two generators, ω and $\mu := \mu_{12}$, by dropping μ_{21} and the relation $\omega\mu_{12}\omega^{-1} = \mu_{21}$. This gives³³

$$\mathcal{G}_F(\mathbb{RP}^3 \uplus \mathbb{RP}^3) = \langle \omega, \mu : \omega^2 = \mu^2 = 1 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z} \rtimes \mathbb{Z}_2, \quad (4.8)$$

where the last isomorphism follows as above (cf. (4.6, 4.7)). The normal \mathbb{Z} is given by the subgroup of words in ω, μ with an even number of letters. A visualization of the generators ω and μ is attempted in Fig. 8. It relies on the picture developed in Fig. 6, where the region inside the sphere S is identified with prime 1 and the region outside S with prime 2.

Following [3], the set of inequivalent unitary irreducible representations (UIR's) can be determined directly as follows. First observe that under any UIR $\rho : (\omega, \mu) \mapsto (\hat{\omega}, \hat{\mu})$ the algebra generated by $\hat{\omega}, \hat{\mu}$ —we call it the ‘representor algebra’—contains $\hat{\omega}\hat{\mu} + \hat{\mu}\hat{\omega}$ in its center. To verify this, just observe that left multiplication of that element by $\hat{\omega}$ equals right multiplication by $\hat{\omega}$ (use $\hat{\omega}^2 = 1$), and likewise with $\hat{\mu}$. Hence, since ρ is irreducible, this central element must be a multiple of the unit $\hat{1}$ (by Schur’s Lemma). This implies that the representor algebra is spanned by $\{\hat{1}, \hat{\omega}, \hat{\mu}, \hat{\omega}\hat{\mu}\}$, i.e. it is four dimensional. Burnside’s theorem (see e.g. § 10 in [83]) then implies that ρ is at most 2-dimensional. There are four obvious one-dimensional UIR’s:

$$\rho_1 : \hat{\omega} = 1, \quad \hat{\mu} = 1, \quad (4.9a)$$

$$\rho_2 : \hat{\omega} = 1, \quad \hat{\mu} = -1, \quad (4.9b)$$

$$\rho_3 : \hat{\omega} = -1, \quad \hat{\mu} = 1, \quad (4.9c)$$

$$\rho_4 : \hat{\omega} = -1, \quad \hat{\mu} = -1. \quad (4.9d)$$

The first two are bosonic the last two fermionic, either of them appears with any of the possible slide symmetries. The two dimensional representations are determined as follows: expand $\hat{\omega}$ and $\hat{\mu}$ in terms of $\{1, \sigma_1, \sigma_2, \sigma_3\}$, where the σ_i are the standard Pauli matrices. That $\hat{\omega}$ and $\hat{\mu}$ each square to $\hat{1}$ means that $\hat{\omega} = \vec{x} \cdot \vec{\sigma}$ and $\hat{\mu} = \vec{y} \cdot \vec{\sigma}$ with $\vec{x} \cdot \vec{x} = \vec{y} \cdot \vec{y} = 1$. Using equivalences we may diagonalize $\hat{\omega}$ so that $\hat{\omega} = \sigma_3$.

³³This shows $(\mathbb{Z}_2 * \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_2 * \mathbb{Z}_2$. This is no surprise. The normal subgroup isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$ of index 2 is given by the set of words in the letters ω, μ_{12} , and μ_{21} containing an even number of the letter ω .

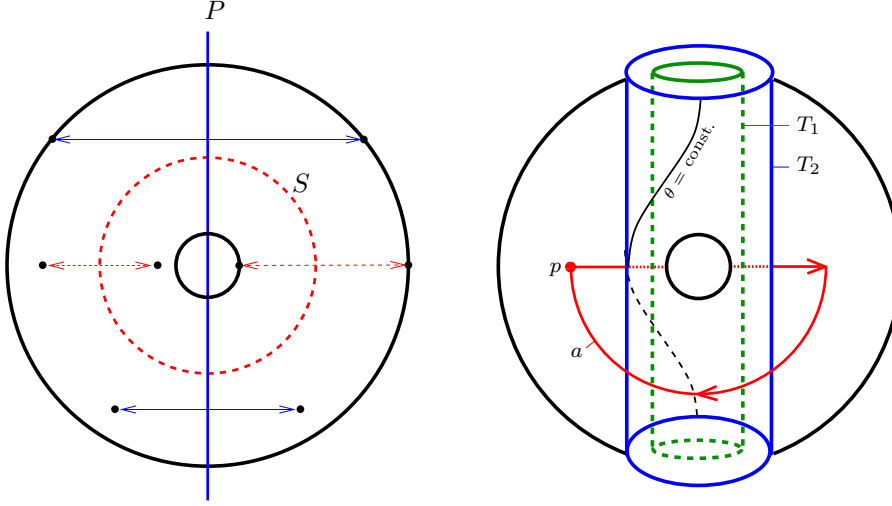


FIGURE 8. The left picture shows how the exchange generator ω can be represented by a combination of two reflections: the first reflection is at the connecting 2-sphere S (dashed), whose action is exemplified by the dashed arrowed lines. This already exchanges the primes, though it is orientation reversing. In order to restore preservation of orientation we add a reflection at a vertical plane, P , whose action is exemplified by the solid arrowed lines. The right picture shows how the slide generator μ may be represented as the transformation of the form (3.8), where the lines ($\theta = \text{const.}$) are now helical with a relative π -rotation between top and bottom, in order to be closed (due to the antipodal identification on S_2). We also draw the generator a for the fundamental group on which the slide acts by conjugating it with b . The straight part pierces both tori whereas the curved part runs in front of them as seen from the observer.

The remaining equivalences are then uniquely fixed by eliminating y_2 and ensuring $y_1 > 0$. Writing $y_1 = \sin \tau$ and $y_3 = \cos \tau$ we thus have :

$$\rho_\tau(\omega) = \hat{\omega} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho_\tau(\mu) = \hat{\mu} = \begin{pmatrix} \cos \tau & \sin \tau \\ \sin \tau & -\cos \tau \end{pmatrix}, \quad \tau \in (0, \pi). \quad (4.10)$$

This continuum of inequivalent 2-dimensional UIRs has interesting properties as regards the statistics types it describes. The angle τ mixes the bosonic and fermionic sector. This mixing is brought about by slides, which physically correspond to transformation where the two ‘lumps of topology’ (geons) truly penetrate each other. Hence these geons have fixed statistics as long they do not come too close (at low energies), but cease to do so when deep scattering is involved. For further

discussion we refer to [2] and especially [80], where the case of more than two geons is considered.

In other non-linear field theories in which to each kink-solution³⁴ there exists an anti-kink solution, general statements can be made regarding spin-statistics correlations [78]. One would not expect such a relation to generalize to gravity without further specifications, since there is no such thing as an anti-geon. This follows immediately from the uniqueness of the prime decomposition.

It has been verified that spinorial manifolds do in fact give rise to half-integer angular momentum states, e.g. in the kinematical Hilbert space of loop quantum gravity [4] (see also [73] for fractional spin in 2+1 dimensions). Hence a natural question is whether the existence or non-existence of spin-statistics violating states throws some light on the different schemes for the construction of states in quantum gravity. This has been looked at by Dowker & Sorkin [12], who showed that the sum-over-histories approach excludes fermionic quantization of lens spaces (which are non-spinorial, as we have seen). Quite generally, they argue that topology changing amplitudes are necessary in order to avoid an embarrassing abundance of sectors, most of which are presumably unphysical. In [13] the same authors discuss rules for assigning weights to individual histories and present three simple conditions of when to assign same weights that suffice to ensure the normal spin-statistics correlation. Unfortunately these rules seem too restrictive, in that they enforce the weights to exclusively come from *abelian* representations of the mapping-class groups. This may altogether rule out spinorial states in the sum-over-histories approach, as it is often the case (any may be generally true) that the mapping-class generated by an overall 2π -rotation is contained in the commutator subgroup and hence annihilated in any abelian representation.³⁵

5. Further remarks on the general structure of $\mathcal{G}_F(\Sigma)$

Generically, the group $\mathcal{G}_F(\Sigma)$ is non-abelian and of infinite order; hence it will not be an easy task to understand its structure. We anticipated that the space of inequivalent UIRs label sectors in quantum gravity. However, this seems to only make sense if the group $\mathcal{G}_F(\Sigma)$ is of type I (see [57, 58]), since only then can we *uniquely* decompose a unitary representation into irreducibles. It is known that a countable discrete group is of type I iff it contains an abelian normal subgroup of finite index [81]. This was indeed the case in our example above, where $\mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z} \rtimes \mathbb{Z}_2$, hence \mathbb{Z} is normal and of finite index. But, generically, being type I will be rather exceptional.

Another important point is the following: suppose we argue (as e.g. the authors of [80] do) that ‘internal’ state spaces should be finite dimensional, in which

³⁴We avoid the name soliton since we do not wish to imply that these objects are dynamically stable.

³⁵This can be explicitly checked for spherical primes and the torus; see Section 4 in [26]. Interestingly, the situation in 2+1 dimensions is quite different, as also discussed there.

case we would only be interested in finite dimensional representations of $\mathcal{G}_F(\Sigma)$. Are these sufficient to ‘make use’ of each element of $\mathcal{G}_F(\Sigma)$? In other words: is any non-trivial element $\mathcal{G}_F(\Sigma)$ non-trivially represented in some finite dimensional representation? If not, the intersection of all kernels of finite dimensional representations would lead to a non-trivial normal subgroup and instead of $\mathcal{G}_F(\Sigma)$ we would only ‘see’ its corresponding quotient. This question naturally leads to the general notion of residual finiteness.

Definition 5.1. A group G is *residually finite* iff for any non-trivial g in G there is a homomorphism ρ_g into a finite group F such that $\rho_g(g)$ is non-trivial in F . Equivalently, for each non-trivial g in G there exists a normal subgroup N_g of G (the kernel of ρ_g) of finite index such that $g \notin N_g$.

Residual finiteness is carried forward by various constructions. For example:

- 1.) A subgroup of a residually finite group is residually finite. The proof is elementary and given in the appendix (Proposition 6.1).
- 2.) Let G be the free product $G = G_1 * \cdots * G_n$. Then G is residually finite iff each G_i is. For a proof see e.g. [36].
- 3.) Let G be finitely generated. If G contains a residually finite subgroup of finite index then G is itself residually finite. The proof is given in the appendix (Proposition 6.6).
- 4.) Let G be finitely generated and residually finite. Then $\text{Aut}(G)$ is residually finite. Again a proof is given in the appendix (Proposition 6.7).

Note that 3.) implies that finite *upward* extensions of residually finite groups (which are the normal subgroups) are residually finite. But, unfortunately, it is *not* likewise true that finite *downward* extensions (i.e. now the finite group is the normal subgroup) of residually finite groups are always residually finite (see e.g. [49]), not even if the extending group is as simple as \mathbb{Z}_2 .³⁶

We already mentioned above that if a group is residually finite the set of finite dimensional representations, considered as functions on the group, separate the group. Many other useful properties are implied by residual finiteness. For example, proper quotients of a residually finite group G are never isomorphic to G . In other words: any surjective homomorphism of G onto G is an isomorphism (such groups are called ‘Hopfian’). Most importantly, any residually finite group has a solvable word problem; see Proposition 6.8 in the appendix.

Large classes of groups share the property of residual finiteness. For example, all free groups are residually finite. Moreover, any group that has a faithful finite-dimensional representation in $GL(n, \mathbb{F})$, where \mathbb{F} is a commutative field, i.e. any

³⁶A simple (though not finitely generated) example is the central product of countably infinite copies of the 8 element dihedral group $D_8 := \langle a, b : a^4 = b^2 = (ab)^2 = e \rangle \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_2$, which can be thought of as the symmetry group of a square, where a is the generator of the \mathbb{Z}_4 of rotations and b is a reflection. Its center is isomorphic to \mathbb{Z}_2 and generated by a^2 , the π -rotation of the square. In the infinite central product, where all centers of the infinite number of copies are identified to a single \mathbb{Z}_2 , every normal subgroup of finite index contains the center. I thank Otto Kegel for pointing out this example.

matrix group over a commutative field, is residually finite. On the other hand, it is also not difficult to define a group that is *not* residually finite. A famous example is the group generated by two symbols a, b and the single relation $a^{-1}b^2a = b^3$. Generally speaking, there are strong group-cohomological obstructions against residual finiteness [50]. We refer to [59] for an introductory survey and references on residual finiteness.

Now we recall that the mapping class group $\mathcal{G}_F(\Sigma)$ is a finite downward extension of $h_F(\text{Aut}(\pi_1(\Sigma)))$, where $\pi_1(\Sigma) = \pi_1(P_1) * \cdots * \pi_1(P_N)$. Suppose each $\pi_1(P_i)$ ($1 \leq i \leq N$) is residually finite, then so is $\pi_1(\Sigma)$ by 2.), $\text{Aut}(\pi_1(\Sigma))$ by 4.), and $h_F(\text{Aut}(\pi_1(\Sigma)))$ by 1.). So we must ask: are the fundamental groups of prime 3-manifolds residually finite? They trivially are for spherical space forms and the handle. For the fundamental group of Haken manifolds residual finiteness has been shown in [45]. Hence, by 3.), it is also true for 3-manifolds which are virtually Haken, i.e. finitely covered by Haken manifolds. As already stated, it is conjectured that every irreducible 3-manifold with infinite fundamental group is virtually Haken. If this were the case, this would show that all prime 3-manifolds, and hence all 3-manifolds³⁷, have residually finite fundamental group.

Assuming the validity of the ‘virtually-Haken’ conjecture (or else discarding those primes which violate it) we learn that $h_F(\text{Aut}(\pi_1(\Sigma)))$ is residually finite. This almost proves residual finiteness for $\mathcal{G}_F(\Sigma)$ in case our primes also satisfy the homotopy-implies-isotopy property, since then $\mathcal{G}_F(\Sigma)$ is just a (downward) central $\mathbb{Z}_2^{n_s+m}$ extension of $h_F(\text{Aut}(\pi_1(\Sigma)))$, where n_s is the number of spinorial primes and m the number of handles. It can be shown that the handle twists (right picture in Fig. 2, which account for m of the \mathbb{Z}_2 s combine with $h_F(\text{Aut}(\pi_1(\Sigma)))$ into a semi-direct product (the extension splits with respect to \mathbb{Z}_2^m). Hence the result is residually finite by 3.). On the other hand, the remaining extension by the neck-twists (left picture in Fig. 2) certainly does not split and I do not know of a general argument that would also show residual finiteness in this case, though I would certainly expect it to hold.

6. Summary and outlook

We have seen that mapping-class groups of 3-manifolds enter naturally in the discussion of quantum general relativity and, more generally, in any diffeomorphism invariant quantum theory. Besides being a fascinating mathematical topic in its own right, there are intriguing aspects concerning the physical implications of diffeomorphism invariance in the presence of non-trivial spatial topologies. Everything we have said is valid in any canonical approach to quantum gravity, may it be formulated in metric or loop variables. These approaches use 3-manifolds of fixed topology as fundamental entities out of which spaces of states and observables are to be built in a diffeomorphism (3-dimensional) equivariant way. Neither the spatial topology nor the spatial diffeomorphisms are replaced by anything discrete or

³⁷Recall that we restrict to compact and orientable(the latter being inessential here) manifolds.

quantum. Therefore the rich structures of mapping-class groups are carried along into the quantum framework. John Friedman and Rafael Sorkin were the first to encourage us to take this structure seriously from a physical perspective. Their work remind us on the old idea of Clifford, Riemann, Tait, and others, that otherwise empty space has enough structure to define localized matter-like properties: quasiparticles out of lumps of topology, an idea that was revived in the 1950s and 60s by John Wheeler and collaborators [84].

The impact of diffeomorphism invariance is one of the central themes in all approaches to quantum gravity. The specific issue of mapping-class groups is clearly just a tiny aspect of it. But this tiny aspect serves very well to give an idea of the range of possible physical implications, which is something that we need badly in a field that, so far, is almost completely driven by formal concepts. For example, the canonical approach differs in its wealth of sectors, deriving from 3-dimensional mapping classes, from the sum-over-histories approach. In the latter, the mapping classes of the bounding 3-manifold do not give rise to extra sectors if they are annihilated after taking the quotient with respect to the normal closure of those diffeomorphisms that extend to the bulk; see e.g. [37][32]. Other examples are the spin-statistics violating sectors, which have been shown to disappear in the sum-over-histories approach in specific cases [12][13]. However, whether the wealth of sectors provided by the canonical approach does indeed impose an ‘embarrassment of richness’ from a physical point of view remains to be seen.

Appendix: Elements of residual finiteness

For the readers convenience this appendix collects some of the easier proofs for the standard results on residual finiteness that where used in the main text. We leave out the proof for the result that a free product is residually finite iff each factor is, which is too involved to be presented here. The standard reference is [36].

In the following $H < G$ or $G > H$ indicates that H is a subgroup of G and $H \triangleleft G$ or $G \triangleright H$ that H is normal. The order of the group G is denoted by $|G|$ and the index of H in G by $[G : H]$. The group identity will usually be written as e . The definition of residual finiteness was already given in Definition 5.1, so that we can start with the first.

Proposition 6.1. *A subgroup of a residually finite group is again residually finite.*

Proof. Let G be residually finite and $G' < G$. Hence, for $e \neq g' \in G'$ there exists a $K \triangleleft G$ of finite index such that $g' \notin K$. Then $K' := K \cap G'$ is clearly a normal subgroup of G' which does not contain g' . It is also of finite index in G' since the cosets of K' in G' are given by the intersections of the cosets of K in G with G' . To see the latter, note that for $g' \in G'$ one has $g'K' = g'(K \cap G') = (g'K) \cap G'$, since $g'k \in G'$ iff $k \in G'$. \square

Lemma 6.2. *Let G be a group and H_i , $i = 1, \dots, n$ a finite number of subgroups of finite indices. Then the intersection $H := \bigcap_{i=1}^n H_i$ is itself of finite index.*

Proof. It suffices to prove this for two subgroups H_1 and H_2 . We consider the left cosets of H , H_1 and H_2 and set $|G/H_i| = n_i$ for $i = 1, 2$. Elements $g, h \in G$ are in the same H -coset iff $h^{-1}g \in H$, which is equivalent to $h^{-1}g \in H_i$ for $i = 1, 2$. Hence the H -cosets are given by the non-trivial intersections of the H_1 -cosets with the H_2 -cosets, of which there are at most $n_1 \cdot n_2$. \square

Lemma 6.3. *Let G be finitely generated group, then the number of subgroups of a given finite index, say n , is finite.*

Proof. We essentially follow Chapter III in [5]. Let $H < G$ be of index n and let $\rho(1), \dots, \rho(n)$ a complete set of left-coset representatives, where without loss of generality we may choose $\rho(1) \in H$. The left cosets are then denoted by $\rho(i)H$ for $i = 1, \dots, n$. We now consider the (H -dependent) homomorphism $\varphi : G \rightarrow S_n$, $g \mapsto \varphi_g$, of G into the symmetric group of degree n , defined through $g(\rho(i)H) =: \rho(\varphi_g(i))H$. Note that this just corresponds to the usual left action of G on the left cosets, which we identified with the numbers $1, \dots, n$ via the choice of coset representatives. Since $gH = H \Leftrightarrow g \in H$ we have $\text{stab}_\varphi(1) := \{g \in G : \varphi_g(1) = 1\} = H$. Now suppose $H' < G$ is also of index n . Repeating the construction above with left-coset representatives $g'(1), \dots, g'(n)$ of H' , where $g'(1) \in H'$, we obtain another homomorphism $\varphi' : G \rightarrow S_n$ with $\text{stab}_{\varphi'}(1) = H'$. Hence $H \neq H' \Rightarrow \varphi \neq \varphi'$. But since G can be generated by a finite number of elements, say m , there are at most $(n!)^m$ different homomorphisms of G into S_n , and hence at most $(n!)^m$ different subgroups of G with index n . \square

For the following we recall that a subgroup $H < G$ is called *characteristic* iff it is invariant under *any* automorphism of G . Note that in case the group allows for non-trivial outer automorphisms this is a strictly stronger requirement than that of *normality* which just requires invariance under inner automorphisms. We define

$$G_n := \bigcap \{H < G : [G : H] = n\}, \quad (6.1a)$$

$$\bar{G}_n := \bigcap \{H < G : [G : H] \leq n\}, \quad (6.1b)$$

i.e., the intersections of all subgroups of index n or index $\leq n$ respectively. Lemma 6.3 ensures that there are only finitely many groups to intersect and Lemma 6.2 guarantees that the intersection is again a group of finite index. Moreover, since an automorphism maps a subgroup of index n to a subgroup of index n it also leaves invariant the sets of subgroups of index n or index $\leq n$ respectively. Hence we have

Lemma 6.4. *G_n and \bar{G}_n are characteristic subgroups of finite index.*

This can be used to give a convenient alternative characterization of residual finiteness for finitely generated groups:

Proposition 6.5. *Let G be finitely generated. G is residually finite \Leftrightarrow*

$$\hat{G} := \bigcap \{H < G : [G : H] < \infty\} = \{e\} \quad (6.2)$$

Proof. “ \Rightarrow ”: Residual finiteness implies that $g \neq e$ in G is not contained in some normal subgroup of finite index. Hence it is not contained in \hat{G} .

“ \Leftarrow ”: For $g \neq e$ we have $g \notin \hat{G}$ and hence $g \notin G_n$ for some n . Since G_n is in particular normal and of finite index, G is residually finite. \square

The following proposition is a conditional converse to Proposition 6.1:

Proposition 6.6. *Let G be finitely generated. If G contains a residually finite subgroup G' of finite index then G is itself residually finite.*

Proof. Let $[G : G'] = k$, then $\bar{G}_n \subseteq G'$ for all $n \geq k$. Hence

$$\hat{G} = \bigcap \{H < G : [G : H] < \infty\} = \bigcap \{H < G' : [G' : H] < \infty\} = \{e\}, \quad (6.3)$$

where in the last step we applied Proposition 6.5 to G' . This is allowed if G' is finitely generated, which is indeed the case since G is finitely generated and G' is of finite index. \square

Note that this proposition implies that finite upward extensions (cf. footnote 16) of residually finite groups are again residually finite. This is not true for finite downward extensions (see below).

There are no analogs of Propositions 6.1 and 6.6 for quotient groups. First, quotient groups of residually finite groups need not be residually finite. An example is provided by the free group F_2 on two generators, which is residually finite (as is any free group F_n , cf. [36]), but not its quotient group $\langle a, b : ab^2a^{-1} = b^3 \rangle$ (see [59], p. 307-308). Second, finite downward extensions of residually finite groups also need not be residually finite [49].

We now turn to other important instances where residual finiteness is inherited.

Proposition 6.7. *Let G be a finitely generated and residually finite group, then $\text{Aut}(G)$ is residually finite.*

Proof. We follow Section 6.5 of [60]. Assuming $\text{Aut}(G)$ is non-trivial, let α be a non-trivial automorphism. Hence there exists a $g_* \in G$ such that $h := g_*^{-1}\alpha(g_*) \neq e$. Residual finiteness of G implies the existence of a $K \triangleleft G$ of finite index, say n , not containing h , so that $h \notin G_n$ (cf. (6.1a)). On the other hand, since G_n is characteristic, we have a natural homomorphism $\sigma : \text{Aut}(G) \rightarrow \text{Aut}(G/G_n)$, simply given by $\sigma(\varphi)(gG_n) := \varphi(g)G_n$, with kernel

$$\text{kernel}(\sigma) = \{\varphi \in \text{Aut}(G) : g^{-1}\varphi(g) \in G_n \forall g \in G\}. \quad (6.4)$$

By Lemma 6.4 G/G_n is finite, and so $\text{Aut}(G/G_n)$ and $\text{image}(\sigma) = \text{Aut}(G)/\text{kernel}(\sigma)$ are finite, too. $h \notin G_n$ now implies $\alpha \notin \text{kernel}(\sigma)$. Hence $\text{Aut}(G/G_n)$ is the sought for finite homomorphic image of $\text{Aut}(G)$ into which α maps non-trivially via σ . \square

Finally we mention one of the most important consequences of residual finiteness:

Proposition 6.8. *Let G be a finitely presented residually finite group, then it has a soluble word problem.*

Proof. The idea is to construct two Turing machines, T_1 and T_2 , which work as follows: Given a word w on the finite set of generators, T_1 simply checks all consequences of the finite number of relations and stops if w is transformed into e . So if w indeed defines the neutral element in G , T_1 will eventually stop. In contrast, T_2 is now so constructed that it stops if w is *not* the neutral element. Using residual finiteness, this is possible as follows: T_2 writes down the image of w under all homomorphisms of G into all finite groups and stops if this image is not trivial. To do this it lists all finite groups and all homomorphisms into them in a ‘diagonal’ (Cantor) fashion. To list all homomorphisms it lists all mappings from the finite set of generators of G into that of the finite group, checking each time whether the relations are satisfied (i.e. whether the mapping defines a homomorphism). If w does not define the neutral element, we know by residual finiteness that it will have a non-trivial image in some finite group and hence T_2 will stop after a finite number of steps. Now we run T_1 and T_2 simultaneously. After a finite number of steps either T_1 or T_2 will stop and we will know whether w defines the neutral element in G or not. \square

References

- [1] Luis Alvarez-Gaumé and Paul Ginsparg. The structure of gauge and gravitational anomalies. *Annals of Physics*, 161:423–490, 1985.
- [2] Charilaos Aneziris et al. Aspects of spin and statistics in general covariant theories. *International Journal of Modern Physics A*, 14(20):5459–5510, 1989.
- [3] Charilaos Aneziris et al. Statistics and general relativity. *Modern Physics Letters A*, 4(4):331–338, 1989.
- [4] Matthias Arnsdorf and Raquel S. Garcia. Existence of spinorial states in pure loop quantum gravity. *Classical and Quantum Gravity*, 16:3405–3418, 1999.
- [5] Gilbert Baumslag. *Topics in Combinatorial Group Theory*. Lectures in Mathematics - ETH Zürich. Birkhäuser, Basel, 1993.
- [6] Robert Beig and Niall Ó Murchadha. The Poincaré group as symmetry group of canonocal general relativity. *Annals of Physics (NY)*, 174:463–498, 1987.
- [7] Theodor Bröcker and Klaus Jänich. *Einführung in die Differentialtopologie*. Springer Verlag, Berlin, 1973.
- [8] Alejandro Cabo and Domingo Louis-Martinez. On Dirac’s conjecture for Hamiltonian systems with first-class constraints. *Physical Review D*, 42(8):2726–2736, 1990.
- [9] John H. Conway et al. *ATLAS of Finite Groups*. Oxford University Press, Oxford, 1985.
- [10] Bryce S. DeWitt. Quantum theory of gravity. I. The canonical theory. *Physical Review*, 160(5):1113–1148, 1967.
- [11] Paul Dirac. *Lectures on Quantum Mechanics*. Belfer Graduate School of Science, Monographs Series Number Two. Yeshiva University, New York, 1964.
- [12] Fay Dowker and Rafael Sorkin. A spin-statistics theorem for certain topological geons. *Classical and Quantum Gravity*, 15:1153–1167, 1998.
- [13] Fay Dowker and Rafael Sorkin. Spin and statistics in quantum gravity. In R.C. Hilborn and G.M. Tino, editors, *Spin-Statistics Connections and Commutation Relations: Experimental Tests and Theoretical Implications*, pages 205–218. American Institute of Physics, New York, 2000.
- [14] Arthur E. Fischer. The theory of superspace. In M. Carmeli, S.I. Fickler, and L. Witten, editors, *Relativity*, Proceedings of the Relativity Conference in the Midwest, Cincinnati, Ohio, June 2-6 1969, pages 303–357. Plenum Press, New York, 1970.
- [15] Arthur E. Fischer. Resolving the singularities in the space of Riemannian geometries. *Journal of Mathematical Physics*, 27:718–738, 1986.
- [16] D.I. Fouxé-Rabinovitch. Über die Automorphismengruppen der freien Produkte I. *Matematicheskii Sbornik*, 8:265–276, 1940.
- [17] D.I. Fouxé-Rabinovitch. Über die Automorphismengruppen der freien Produkte II. *Matematicheskii Sbornik*, 9:297–318, 1941.
- [18] Charles Freifeld. One-parameter subgroups do not fill a neighborhood of the identity in an infinite-dimensional lie (pseudo-) group. In Cecile M. DeWitt and John A. Wheeler, editors, *Battelle Rencontres*, 1967 Lectures in Mathematics and Physics, pages 538–543. W.A. Benjamin, New York, 1968.
- [19] John Friedman and Rafael Sorkin. Spin 1/2 from gravity. *Physical Review Letters*, 44:1100–1103, 1980.

- [20] John Friedman and Donald Witt. Homotopy is not isotopy for homeomorphisms of 3-manifolds. *Topology*, 25(1):35–44, 1986.
- [21] Gary W. Gibbons. The elliptic interpretation of black holes and quantum mechanics. *Nuclear Physics*, B 98:497–508, 1986.
- [22] Nick D. Gilbert. Presentations of the automorphisms group of a free product. *Proceedings of the London Mathematical Society*, 54:115–140, 1987.
- [23] Domenico Giulini. On the possibility of spinorial quantization in the skyrme model. *Modern Physics Letters A*, 8(20):1917–1924, 1993.
- [24] Domenico Giulini. 3-manifolds for relativists. *International Journal of Theoretical Physics*, 33:913–930, 1994.
- [25] Domenico Giulini. Diffeomorphism invariant states in Witten’s 2+1 quantum gravity on $R \times T^2$. *Classical and Quantum Gravity*, 12:2735–2745, 1995.
- [26] Domenico Giulini. On the configuration-space topology in general relativity. *Helvetica Physica Acta*, 68:86–111, 1995.
- [27] Domenico Giulini. Quantum mechanics on spaces with finite fundamental group. *Helvetica Physica Acta*, 68:439–469, 1995.
- [28] Domenico Giulini. What is the geometry of superspace? *Physical Review D*, 51(10):5630–5635, 1995.
- [29] Domenico Giulini. The group of large diffeomorphisms in general relativity. *Banach Center Publications*, 39:303–315, 1997.
- [30] Domenico Giulini. On the construction of time-symmetric black hole initial data. In F.W. Hehl, C. Kiefer, and R. Metzler, editors, *Black Holes: Theory and Observation*, volume 514 of *Lecture Notes in Physics*, pages 224–243. Springer Verlag, Berlin, 1998.
- [31] Domenico Giulini. That strange procedure called quantisation. In D. Giulini, C. Kiefer, and C. Lämmerzahl, editors, *Quantum Gravity: From Theory to Experimental Search*, volume 631 of *Lecture Notes in Physics*, pages 17–40. Springer Verlag, Berlin, 2003.
- [32] Domenico Giulini and Jorma Louko. Theta-sectors in spatially flat quantum cosmology. *Physical Review D*, 46:4355–4364, 1992.
- [33] Alfred S. Goldhaber. Connection of spin and statistics for charge-monopole composites. *Physical Review Letters*, 36(19):1122–1125, 1976.
- [34] Mark J. Gotay. On the validity of Dirac’s conjecture regarding first-class secondary constraints. *Journal of Physics A: Mathematical and General*, 16:L141–L145, 1983.
- [35] Mark J. Gotay and James M. Nester. Apartheid in the Dirac theory of constraints. *Journal of Physics A: Mathematical and General*, 17:3063–3066, 1984.
- [36] Karl W. Grünberg. Residual properties of infinite soluble groups. *Proceedings of the London Mathematical Society*, 7:29–62, 1957.
- [37] James Hartle and Donald Witt. Gravitational θ -states and the wave function of the universe. *Physical Review D*, 37(10):2833–2837, 1988.
- [38] Peter Hasenfratz and Gerard ’t Hooft. Fermion-boson puzzle in a gauge theory. *Physical Review Letters*, 36(19):1119–1122, 1976.
- [39] Allan E. Hatcher. Linearization in 3-dimensional topology. In Olli Lehto, editor, *Proceedings of the International Congress of Mathematicians*, volume 1, pages 463–468, Helsinki, 1978. American Mathematical Society (1980).

- [40] Allen E. Hatcher. Notes on basic 3-manifold topology. Online available at www.math.cornell.edu/~hatcher/3M/3Mdownloads.html.
- [41] Allen E. Hatcher. Homeomorphisms of sufficiently large P^2 -irreducible 3-manifolds. *Topology*, 15:343–347, 1976.
- [42] Allen E. Hatcher. A proof of the Smale conjecture, $\text{Diff}(S^3) \simeq O(4)$. *Annals of Mathematics*, 117:553–607, 1983.
- [43] Stephen W. Hawking and George F.R. Ellis. *The Large Scale Structure of Spacetime*. Cambridge University Press, Cambridge, 1973.
- [44] John Hempel. *3-Manifolds*. Princeton University Press, Princeton, New Jersey, 1976.
- [45] John Hempel. Residual finiteness for 3-manifolds. In S.M. Gersten and J.R. Stallings, editors, *Combinatorial Group Theory and Topology*, volume 111 of *Annals of Mathematics Studies*, pages 379–396. Princeton University Press, Princeton, New Jersey, 1987.
- [46] Harrie Hendriks. Application de la théorie d’obstruction en dimension 3. *Mémoires de la Société Mathématique de France*, 53:81–196, 1977. Online available at www.numdam.org.
- [47] Harrie Hendriks and Darryl McCullough. On the diffeomorphism group of a reducible manifold. *Topology and its Applications*, 26:25–31, 1987.
- [48] Marc Henneaux and Claudio Teitelboim. *Quantization of Gauge Systems*. Princeton University Press, Princeton, New Jersey, 1992.
- [49] Paul R. Hewitt. Extensions of residually finite groups. *Journal of Algebra*, 163(1):757–771, 1994.
- [50] Paul R. Hewitt. An army of cohomology against residual finiteness. In C.M. Campbell, T.C. Hurley, E.F. Robertson, S.J. Tobin, and J.J. Ward, editors, *Groups ‘93*, volume 212 of *London Mathematical Society, Lecture Notes*, pages 305–313. Cambridge University Press, Cambridge, 1995.
- [51] Christopher J. Isham. Θ -states induced by the diffeomorphism group in canonically quantized gravity. In J.J. Duff and C.J. Isham, editors, *Quantum Structure of Space and Time*, Proceedings of the Nuffield Workshop, August 3–21 1981, Imperial College London, pages 37–52. Cambridge University Press, London, 1982.
- [52] John Kalliongis and Darryl McCullough. Isotopies of 3-manifolds. *Topology and its Applications*, 71(3):227–263, 1996.
- [53] Claus Kiefer. *Quantum Gravity*, volume 124 of *International Series of Monographs on Physics*. Clarendon Press, Oxford, 2004.
- [54] Hellmuth Kneser. Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten. *Jahresberichte der deutschen Mathematiker Vereinigung*, 38:248–260, 1929.
- [55] Martin D. Kruskal. Maximal extension of schwarzschild metric. *Physical Review*, 119(5):1743–1745, 1960.
- [56] Jorma Louko. Single-exterior black holes. In J. Kowalski-Glikman, editor, *Towards Quantum Gravity*, volume 541 of *Lecture Notes in Physics*, pages 188–202. Springer Verlag, Berlin, 2000.
- [57] George W. Mackey. Infinite-dimensional group representations. *Bulletin of the American Mathematical Society*, 69:628–686, 1963.

- [58] George W. Mackey. *Unitary Group Representations in Physics, Probability, and Number Theory*. Advanced Book Classics. Addison-Wesley Publishing Company, Redwood City, California, 1989. Originally published in 1978 as part of the Mathematics Lecture Notes Series by the Benjamin/Cummings Publishing Company.
- [59] Wilhelm Magnus. Residually finite groups. *Bulletin of the American Mathematical Society*, 75(2):305–316, 1969.
- [60] Wilhelm Magnus, Abraham Karrass, and Donald Solitar. *Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations*, volume XIII of *Pure and Applied Mathematics*. Interscience Publishers, John Wiley and Sons Inc., New York, 1966. There is a Dover reprint edition of this book.
- [61] G.S. McCarty. Homeotopy groups. *Transactions of the American Mathematical Society*, 106:293–303, 1963.
- [62] Darryl McCullough. Topological and algebraic automorphisms of 3-manifolds. In Renzo Piccinini, editor, *Groups of Homotopy Equivalences and Related Topics*, volume 1425 of *Springer Lecture Notes in Mathematics*, pages 102–113. Springer Verlag, Berlin, 1990.
- [63] Darryl McCullough and Andy Miller. Homeomorphisms of 3-manifolds with compressible boundary. *Memoirs of the American Mathematical Society*, 61(344), 1986.
- [64] Dusa McDuff. The lattice of normal subgroups of the group of diffeomorphisms or homeomorphisms of an open manifold. *Journal of the London Mathematical Society*, 18 (2. series):353–364, 1978.
- [65] John W. Milnor. Groups which act on S^n without fixed points. *American Journal of Mathematics*, 79(3):623–630, 1957.
- [66] John W. Milnor. A unique decomposition theorem for 3-manifolds. *American Journal of Mathematics*, 84(1):1–7, 1962.
- [67] John W. Milnor. *Introduction to Algebraic K-Theory*, volume 72 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, New Jersey, 1971.
- [68] Charles Misner and John A. Wheeler. Classical physics as geometry: Gravitation, electromagnetism, unquantized charge, and mass as properties of curved empty space. *Annals of Physics (NY)*, 2:525–660, 1957.
- [69] Philip Nelson and Luis Alvarez-Gaumé. Hamiltonian interpretation of anomalies. *Communications in Mathematical Physics*, 99(1):103–114, 1985.
- [70] Steven Plotnick. Equivariant intersection forms, knots in S^4 and rotations in 2-spheres. *Transactions of the American Mathematical Society*, 296(2):543–575, 1986. Online available at www.jstor.org.
- [71] Kurt Reidemeister. Homotopieringe und Linsenräume. *Abhandlungen des Mathematischen Seminars der Universität Hamburg*, 11:102–109, 1935.
- [72] Wolfgang Rindler. Elliptic Kruskal-Schwarzschild space. *Physical Review Letters*, 15(26):1001–1002, 1965.
- [73] Joseph Samuel. Fractional spin from gravity. *Physical Review Letters*, 71(2):215–218, 1993.
- [74] Herbert Seifert. Topologie dreidimensionaler gefaseter Räume. *Acta Mathematica*, 60:147–288, 1933.

- [75] Tony Hilton Royle Skyrme. Kinks and the Dirac equation. *Journal of Mathematical Physics*, 12(8):1735–1743, 1971.
- [76] Stephen Smale. Diffeomorphisms of the 2-sphere. *Proceedings of the American Mathematical Society*, 10(4):621–626, 1959.
- [77] Rafael Sorkin. Introduction to topological geons. In P.G. Bergmann and V. De Sabbata, editors, *Topological Properties and Global Structure of Space-Time*, volume B138 of *NATO Advanced Study Institutes Series*, page 249. D. Reidel Publishing Company, Dordrecht-Holland, 1986.
- [78] Rafael Sorkin. A general relation between kink-exchange and kink-rotation. *Communications in Mathematical Physics*, 115:421–434, 1988.
- [79] Rafael Sorkin. Classical topology and quantum phases: Quantum geons. In S. De Filippo, M. Marinaro, G. Marmo, and G. Vilasi, editors, *Geometrical and Algebraic Aspects of Nonlinear Field Theory*, pages 201–218. Elsevier Science Publishers B.V., Amsterdam, 1989.
- [80] Rafael Sorkin and Sumati Surya. An analysis of the representations of the mapping class group of a multi-geon three-manifold. *International Journal of Modern Physics A*, 13(21):3749–3790, 1998.
- [81] Elmar Thoma. Über unitäre Darstellungen abzählbarer, diskreter Gruppen. *Mathematische Annalen*, 153(2):111–138, 1964. Online available via www.digizeitschriften.de.
- [82] Charles B. Thomas. *Elliptic Structures on 3-Manifolds*, volume 104 of *London Mathematical Society Lecture Notes Series*. Cambridge University Press, Cambridge, 1986.
- [83] Hermann Weyl. *Gruppentheorie und Quantenmechanik*. Wissenschaftliche Buchgesellschaft, Darmstadt, 1981.
- [84] John A. Wheeler. *Geometrodynamics*. Academic Press, New York, 1962.
- [85] John Henry Constantine Whitehead. On incidence matrices, nuclei and homotopy types. *Annals of Mathematics*, 42(5):1197–1239, 1941.
- [86] Donald Witt. Symmetry groups of state vectors in canonical quantum gravity. *Journal of Mathematical Physics*, 27(2):573–592, 1986.
- [87] Donald Witt. Vacuum space-times that admit no maximal slices. *Physical Review Letters*, 57(12):1386–1389, 1986.
- [88] Joseph A. Wolf. *Spaces of Constant Curvature*. MacGraw-Hill Book Company, New York, 1967.

Domenico Giulini
 Physikalisches Institut
 Universität Freiburg
 Hermann-Herder-Straße 3
 D-79104 Freiburg
 Germany
 e-mail: giulini@physik.uni-freiburg.de